Problem 1: Let $f, g$ be functions whose graphs are provided below.

1. Where do $f', g'$ exist?

   Solution: $f'(x)$ exists everywhere $f$ is defined except for $x \in \{-1, 1, 3\}$. $g'(x)$ exists everywhere $g$ is defined.

2. Where is $f(x)$ increasing/decreasing? Where is $f'(x)$ positive/negative? Same questions for $g$.

   Solution: $f$ is increasing on $(-5, 1)$ and $(3, 5)$; $f'$ is positive on those intervals.

3. Find $f'(-3)$ and $g'(-3)$

   Solution: $f'(-3) = 1/2$, and $g'(-3) = 2$, as these are the respective slopes.

4. If $F(x) = f(x)/g(x)$, $F'(-3) =$

   Solution: $F'(-3) = \frac{f'(-3)g(-3) - g'(-3)f(-3)}{g(-3)^2} = \frac{\frac{1}{2} \cdot 2 - 2 \cdot -3}{2^2}$

5. Estimate $g'(1.5)$ from the graph. Use it to find the equation of the tangent line.

   Solution: The slope is approximately $-1$, so $g(1.5) \approx -1$. Since $g(1.5) \approx 3.5$, the tangent equation is $-(x - 1.5) + 3.5$. 

6. Sketch \( f'(x) \) [3 points] and \( g'(x) \) [3 points].

**Solution:** For \( f \), the graph of \( f' \) is piecewise-constant, with values \( 1/2, 3, -2 \) and \( 1/2 \) on corresponding intervals. The approximate sketch of \( g' \) is below.
Problem 2: Let \( f(x) = \frac{1}{x+1} \). Using the *definition of derivative only*, find the derivative of \( f(x) \). Note: show all work. No credit will be given for just an answer. You may not use the power rule.

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\frac{1}{x+1+h} - \frac{1}{x+1}}{h}
\]

\[
= \lim_{h \to 0} \frac{(x+1) - (x+1+h)}{h(x+1+h)(x+1)}
\]

\[
= \lim_{h \to 0} \frac{-h}{h(x+1+h)(x+1)}
\]

\[
= \lim_{h \to 0} \frac{-1}{h(x+1+h)(x+1)}
\]

\[
= \frac{-1}{(x+1)^2}.
\]

Problem 3: Let \( f(x) = \ln(x) \). Find \( F'(x) \) using only the chain rule and the fact that \((e^x)' = e^x\).

Solution: By definition, \( e^{\ln(x)} = x \). By the chain rule, \( \frac{d}{dx} e^{\ln(x)} = e^{\ln(x)} (\ln(x))' \).

Differentiating both sides, we get

\[
\frac{d}{dx} e^{\ln(x)} = \frac{d}{dx} x
\]

\[
\Rightarrow e^{\ln(x)} (\ln(x))' = 1
\]

\[
x (\ln(x))' = 1
\]

\[
(\ln(x))' = \frac{1}{x}
\]

\[
F'(x) = \frac{1}{x}.
\]

Problem 4: After taking an exam, you throw your favorite instructor through the window. The altitude of your instructor, in meters, as a function of time, in seconds, is given by the function \( f(t) = 6 - 5(t-1)^2 \).

How fast (in the vertical direction) did you throw your favorite instructor? How fast is your instructor going at the moment of impact with the ground?

Solution: The initial velocity is \( f'(0) = 2 \cdot -5(0 - 1) = 10 \frac{m}{s} \). The moment of impact is when \( f(t) = 0 \); solving for \( t \), we find that \( t \approx 2.0954 \), and \( f'(t) \approx -10.9545 \); that is, \( 10.9545 \frac{m}{s} \) downwards.
Problem 5: Let \( f, g : \mathbb{R} \to \mathbb{R} \) be differentiable functions. The values of \( f(x), f'(x), g(x), g'(x) \) for some values of \( x \) are given in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>( g'(x) )</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>11</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

Using the rules of differentiation, compute the following. **Justify your work!**

1. \( F(x) = f(x) \cdot g(x) \). \( F'(2) = \)

2. \( F(x) = f(x)/g(x) \). \( F'(2) = \)

3. \( F(x) = g(f(x)) \). \( F'(3) = \)

4. \( F(x) = e^{g(x)} \). \( F'(5) = \)

5. \( F(x) = 2f(x) + 3g(x) \). \( F'(5) = \)

6. \( F(x) = \ln(f(x) \cdot g(x)) \). \( F'(1) = \)

7. \( F(x) = \frac{\sin(f(x))}{\cos(g(x) + 1)} \). \( F'(3) = \).

**Sample solutions:** To get 1, note that \( F'(2) = f'(2)g(2) + f(2)g'(2) \) by the product rule, and so \( F'(2) = 5 \cdot 1 + 3 \cdot 4 = 17. \)

To get 4, note that \( F'(5) = e^{g(5)}g'(5) = e^7 \cdot 6 \) by the chain rule.

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Other parts are done in a similar way.

**Problem 6:** Find the following derivatives:

1. \( \frac{d}{dx} \left(x^{1/2} + x^{1/3} + 7\right) = \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} \) (power rule)

2. \( \frac{d}{dx} \left(\frac{\cos(x)}{\sqrt{x^2 + 1}}\right) = -\sin(x)\sqrt{x^2 + 1} - \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x \cdot \cos(x) \) \( \left(\text{quotient, power, chain rule}\right) \)

3. \( \frac{d}{dx} \left(\frac{\sin(x^3)}{e^x}\right) = \frac{3x^2 \cos(x^3) e^x - e^x \sin(x^3)}{e^{2x}} \)

4. \( \frac{d}{dx} \left(\ln(\sin(x) \cdot \cos(x))\right) = \frac{\sin'(x) \cos(x) + \cos'(x) \sin(x)}{\sin(x) \cos(x)} = \frac{\cos^2(x) - \sin^2(x)}{\sin(x) \cos(x)}. \)

Note that we could get the same result by noting that

\[ \ln(\sin(x) \cos(x)) = \ln(\sin(x)) + \ln(\cos(x)). \]

Then \( \frac{d}{dx} (\ln(\sin(x)) + \ln(\cos(x))) = \frac{\cos(x)}{\sin(x)} + \frac{-\sin(x)}{\cos(x)} = \frac{\cos^2(x) - \sin^2(x)}{\sin(x) \cos(x)}. \)
Problem 7: Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function on the real line. Match the limits below to their values.

1. \( \lim_{a \to 0} \frac{f(x + a) - f(x)}{a} \)
2. \( \lim_{a \to 0} \frac{f(2a) - f(-2a)}{4a} \)
3. \( \lim_{a \to 0} \frac{f(a) - f(0)}{2a} \)
4. \( \lim_{a \to 0} \frac{f(0) - f(2a)}{2a} \)
5. \( \lim_{a \to 0} \frac{f(a) - f(0)}{a} \)
6. \( \lim_{a \to 0} \frac{f(x + a) - f(x - a)}{2a} \)
7. \( \lim_{a \to 0} \frac{f(-a) - f(a)}{-a} \)

Key: 
- 1 2 3 4 5 6 7
  - e  a  c  b  a  e  d

Notes: By definition, \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \), which is the expression in 1 (except with the variable \( h \) renamed to \( a \)); thus 1-e.

Similarly, \( f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(0) - f(0)}{h} \), and so 5-a.

But these are not the only expressions for a derivative: the general idea is that

\[
f'(x) = \lim_{\text{difference in argument of } f} \frac{\text{difference in value of } f}{b-a}
\]

as both \( b \) and \( a \) approach \( x \) (and thus the difference in the argument, \( b - a \), approaches 0).

For that reason, \( f'(x) = \lim_{a \to 0} \frac{f(x + a) - f(x - a)}{2a} \) as well (so 6-e), and 2-a: \( f'(0) = \lim_{a \to 0} \frac{f(2a) - f(-2a)}{4a} \).

To get other limits, note that

\[
\lim_{a \to 0} \frac{f(a) - f(0)}{2a} = \frac{1}{2} \lim_{a \to 0} \frac{f(a) - f(0)}{a} = \frac{1}{2} f'(0),
\]

so 3-c; similarly,

\[
\lim_{a \to 0} \frac{f(0) - f(2a)}{2a} = -\lim_{a \to 0} \frac{f(2a) - f(0)}{2a} = -f'(0),
\]

so 4-b, and

\[
\lim_{a \to 0} \frac{f(-a) - f(a)}{-a} = 2 \lim_{a \to 0} \frac{f(a) - f(-a)}{2a} = 2 f'(0),
\]

so 7-d.