Consider a half-sphere of radius $R$, and a solid cylinder of radius $R$ and height $R$, out of which a solid cone with apex at the center of the bottom face has been removed.

We will compute the volume of the sphere by looking to cross-sections parallel to the bases of these solids.

1. (2 points) Find the cross-section area of the half-sphere at height $h$ from the base (as a function of $h$).
   
   **Solution:** The cross-sections are disks of radius $r = \sqrt{R^2 - h^2}$ and area $\pi r^2 = \pi (R^2 - h^2)$.

2. (2 points) Find the cross-section area of the second solid at height $h$ (as a function of $h$).
   
   **Solution:** The cross-section is an annulus (a disk with a hole). The radius of the outer circle is $R$, the radius of the inner circle is $h$, and the area is $\pi (R^2 - h^2)$.

3. (1 point) What is the volume of the second solid?
   
   **Solution:** The volume is $V(\text{cylinder}) - V(\text{cone}) = \pi R^2 \cdot R - \pi R^2 \cdot R \cdot \frac{1}{3} = \frac{2}{3} \pi R^3$.

4. (1 point) Apply Cavalieri’s principle to get the volume of the half-sphere, as a function of $R$.
   
   **Solution:** By Cavalieri’s principle, since the cross-sections are the same, so are the volumes. The volume of the hemisphere is thus $\frac{1}{2} \pi R^3$.

5. (4 points) Find the volume of the sphere of radius 1 using the method of cylindrical shells.
   
   **Solution:** the shells have radius $r = x$ and height $h = \sqrt{1 - x^2}$, so
   
   $$V = \int_{-1}^{1} 2\pi \cdot r(x) \cdot h(x) \, dx$$
   
   $$= \int_{-1}^{1} 2\pi \cdot x \cdot \sqrt{1 - x^2} \, dx$$
   
   $$= 2 \int_{x=0}^{x=1} 2\pi \cdot x \cdot \sqrt{1 - x^2} \, dx$$

   1
Letting $u = 1 - x^2$, $du = -2x \, dx$ we make the substitution:

\[
V = 2 \int_{u(u(0)=1)}^{u(u(1)=0)} -\sqrt{u} \, du
\]

\[
= 2 \int_{0}^{1} \sqrt{u} \, du
\]

\[
= 2 \frac{2}{3} u^{3/2}\bigg|_{0}^{1}
\]

\[
= \frac{4}{3}
\]

which is consistent with the answer for the previous problem.