March 21, 2016

Pb 1

a. Answer parts (a), and (b) for the function \( f(x) = \sqrt{1 + x} \). Find the linear (tangent line) approximation of \( g(x) \) for \( x \) near zero. b. Find the quadratic approximation of \( g(x) \) for \( x \) near zero. c. Use the linear approximation from part (a) above to approximate \( \sqrt{1.02} \).

\[
\begin{align*}
g(x) &= \sqrt{1 + x} \\
a &= 0 \\
 g'(x) &= \frac{1}{2\sqrt{1 + x}} \\
g''(x) &= \frac{-1}{4(1 + x)^{3/2}} \\
\end{align*}
\]

For \( x \approx 0 \), \( \sqrt{1 + x} \approx 1 + \frac{1}{2}x \)  END OF (a)

\[
\begin{align*}
\sqrt{1.02} &\approx 1 + \frac{1}{2}(0.02) = 1 + 0.01 = 1.01 \\
\end{align*}
\]

b. \[
y = g(0) + g'(0)(x-0) + \frac{g''(0)}{2}(x-0)^2
\]
\[
y = 1 + \frac{1}{2}(x-0) + \frac{-1}{8}(x-0)^2
\]

\[
\begin{align*}
g(x) &= (1 + x)^{1/2} \\
g'(x) &= \frac{1}{2}(1 + x)^{-1/2} \\
g''(x) &= -\frac{1}{4}(1 + x)^{-3/2}
\end{align*}
\]

\[
\begin{align*}
y &= 1 + \frac{x}{2} - \frac{x^2}{8}
\end{align*}
\]

Pb 2

Suppose the linear approximation for a function \( f(x) \) at \( a = 3 \) is given by the tangent line \( y = -2x + 10 \). a. What are \( f(3) \) and \( f'(3) \)? b. If \( g(x) = [f(x)]^2 \), find the linear approximation for \( g(x) \) at \( a = 3 \).

\[
\begin{align*}
\text{Lin. Appr. of } f(x) \text{ at } a = 3 \text{ is } y &= -2x + 10 \\
f(3) &= -2(3) + 10 = 4 \\
f'(3) &= -2
\end{align*}
\]

\[
\begin{align*}
\text{If } g(x) = [f(x)]^2, \text{ find lin. appr. of } g \text{ near } 3. \\
g'(x) &= 2f(x)f'(x) \\
\end{align*}
\]

\[
\begin{align*}
g(3) &= [f(3)]^2 = 4^2 = 16 \\
g'(3) &= 2f(3)f'(3) = 2 \cdot 4 \cdot (-2) = -16
\end{align*}
\]

\[
y = -16 -16(x-3)
\]
Suppose \( f \) and \( g \) are differentiable functions. The line \( y = 2 - 3x \) is the linear approximation to \( f \) at \( x = 2 \), and the line \( y = 1 + 2x \) is the linear approximation to \( g \) at \( x = 2 \).

\[
\begin{align*}
g'(2) &= -3 \\
g''(2) &= -4 \\
f'(2) &= 1 \\
f''(2) &= 5 \\
\frac{dy}{dx} &= 2
\end{align*}
\]

a. Find \( f(2), g(2), f'(2), g'(2) \).

b. Let \( (x, y) = (f(x), g(x)) \). Find the linear approximation to the graph of \( f \) at \( x = 2 \).

Find the points on the curve \( x(t) = 2 - t^2 + 2, \ y(t) = t^3 - t \) where the tangent line is horizontal and the points where it is vertical.

\[
\begin{align*}
x(t) &= 2 - t^2 + 2 \\
y(t) &= t^3 - t
\end{align*}
\]

Horizontal when slope is zero:

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t - 1}
\]

This is zero when \( 3t^2 - 3 = 0 \) or \( t^2 = 1 \).

\[
t = \pm 1
\]

\[
x = 1 - 1^2 + 2, \ y = 1 - 1 - 2
\]

Vertical when slope is infinity:

\[
t = \frac{1}{2}
\]

\[
x = \frac{3}{2} + 2, \ y = \frac{1}{2} - \frac{3}{2}
\]

\[
x = \frac{1}{4} - \frac{3}{4} = \frac{3}{4}, \ y = \frac{1 - 12}{8} = -\frac{11}{8}
\]

(\( \frac{3}{4} \), -\( \frac{11}{8} \)) vertical tangent
Pb #6: \( f(x) = \begin{cases} a \ x^2 & x \leq 1 \\ -x^2 + 4x + b & x > 1 \end{cases} \)

a) Find values of \( a \) and \( b \) that make \( f \) differentiable everywhere.

From the left: \( f'(1) = 2ax \bigg|_{x=1} = 2a \)

From the right: \( f'(1) = -2x + 4 + 0 \bigg|_{x=1} = -2 + 4 = 2 \)

\( 2a = 2 \Rightarrow a = 1 \)

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \]

\( a \cdot 1^2 = -1^2 + 4 + b \)

\( 1 = -1 + 4 + b \Rightarrow b = -2 \)

\[ f(x) = \begin{cases} 1 \ x^2 & x \leq 1 \\ -x^2 + 4x - 2 & x > 1 \end{cases} \]

b) Find \( f(1), \ f'(1) \)

\( f(1) = 1 \)

\( f'(x) = \begin{cases} 2x & x \leq 1 \\ -2x + 4 & x > 1 \end{cases} \)

\( f(1) = 2 \)
A tall man with excellent vision whose eye level is 6 ft above the ground walks toward a very small bug on a wall at a rate of 2 ft/s. The bug is 15 ft above the ground. At what rate is the viewing angle changing when the man is 30 ft from the wall?

\[
\tan \theta = \frac{q}{x} = \frac{q}{x(t)} = q\left[\frac{x(t)}{x}\right]^{-1}
\]

Differentiating both sides with respect to time gives:

\[
\sec^2 \theta \frac{d\theta}{dt} = q \left(-\frac{q}{x(t)^2}\right) \frac{dx}{dt}
\]

When \(x = 30\) ft, the hypotenuse is:

\[
\text{hypotenuse} = \sqrt{q^2 + 30^2} = \sqrt{981 + 900} = \sqrt{1881}
\]

Thus, the viewing angle is increasing at a rate of:

\[
\frac{d\theta}{dt} = \frac{18}{900} \cdot \frac{900}{981} = \frac{18}{981} \
\]

The viewing angle is increasing at a rate of \(\frac{2}{109}\) rad/sec when the man is 30 ft away from the wall.
\[ \lim_{\theta \to 0} \frac{5\theta^2}{\sin^2(3\theta)} = ? \]

\[ \lim_{\theta \to 0} \frac{\frac{3\theta}{\sin 3\theta}}{\frac{3\theta}{\sin 3\theta}} = \frac{1}{3} \]

\[ \frac{1}{3} \]

\[ \boxed{\frac{5}{9}} \]

\[ \lim_{t \to 2^+} \left( \frac{t}{4} \right)^{\frac{1}{t-2}} = ? \]

\[ \lim_{t \to 2^+} \left( \frac{t}{4} \right)^{\frac{2}{t-2}} = \frac{2}{0} = -\infty \]

\[ \left( \frac{2}{0} \right)^{\frac{2}{0}} = \infty \]

\[ \lim_{t \to 2^+} \left( \frac{t}{4} \right)^{\frac{2}{t-2}} = \frac{\frac{2}{0}}{\frac{2}{0}} = \frac{\frac{2}{0}}{\frac{2}{0}} = 0 \]

\[ \frac{d}{dx} \left( \frac{e^{2x}}{x^2} \right) = \frac{2e^{2x}x^2 - e^{2x} \cdot 2x}{x^4} \]

\[ = \frac{2x^2e^{2x} - 2xe^{2x}}{x^4} = \frac{2xe^{2x}(x-1)}{x^3} \]

\[ = \frac{2e^{2x}(x-1)}{x^3} \]

\[ \lim_{x \to +\infty} \frac{2e^x - 8e^{-4x}}{5e^x + 3e^{-4x}} = \]

\[ = \lim_{x \to +\infty} \frac{2e^x}{5e^x} = \frac{2}{5} \]

\[ e^x \to \infty \]

\[ e^{-4x} \to 0 \]

The same limit, as \( x \to -\infty \) is \( -\frac{8}{3} \).

\[ \lim_{x \to 0} \frac{\cos^9 \cos x - \sin^9 \sin x}{x} = \cos^9 \]

\[ \lim_{x \to 0} \frac{\cos^9 \cos x - \cos^9}{x} = \sin^9 \sin x \]

\[ \lim_{x \to 0} \frac{\cos^9 (\cos x - 1)}{x} = (\sin^9) \]

\[ \boxed{0} \]

\[ \lim_{x \to 0} \frac{\cos^9 - \cos^9}{x} = \frac{\cos^9}{\cos x + 1} \]

\[ \lim_{x \to 0} \frac{\sin x}{\cos x + 1} \]

\[ \lim_{x \to 0} \frac{\sin x}{\cos x + 1} \]
An upside down conical tank full of water has "base" radius of 5 meters and height of 7 meters. The water is being drained at a rate of 3 cubic meters per minute. Find the rate of change of the height when it (the height) is 4 meters.

\[
\frac{d}{dx} \left( \sin(e^x) + e^{\sqrt{x}} \right) + (\cos(e^x))e^x + e^{\frac{1}{2\sqrt{x}}} = \frac{1}{2}
\]

\[
\text{Problem 9}
\]

\[
V = \frac{1}{3} \pi r^2 \cdot h \rightarrow V(t) = \frac{1}{3} \pi r^2(t) h(t)
\]

\[
\frac{r}{h} = \frac{5}{7} \Rightarrow r(t) = \frac{5}{7} h(t)
\]

\[
V(t) = \frac{\pi}{3} \cdot \frac{25}{49} \cdot h^2(t) h(t)
\]

\[
\frac{d}{dt} V(t) = \frac{25\pi}{147} \cdot h^3(t)
\]

\[
\frac{dV}{dt} = \frac{25\pi}{147} \cdot 3 \cdot h^2(t) \cdot \frac{dh}{dt}
\]

\[
-3 = \frac{25\pi}{147} \cdot 3 \cdot 16 \cdot \frac{dh}{dt}
\]

\[
\frac{dh}{dt} \bigg|_{h=4} = -\frac{147}{25\pi} \cdot \frac{3}{16} \text{ m/min}
\]

The height is decreasing at a rate of \(\frac{147}{25\cdot16\cdot\pi} \text{ m/min}\) when it is 4 m.