Classifying Modular Categories

Eric Rowell Texas A&M University

May 2013

Supported by USA NSF grant DMS1108725

joint works with Richard Ng, Paul Bruillard, Zhenghan Wang and César Galindo

Modular Categories

Definition

A modular category $\mathcal{C}$ is a braided fusion category (BFC) with a non-degenerate ribbon structure.

Remark

- Simple isomorphism classes of objects $\{X_0 = \mathbf{1}, X_1, \ldots, X_{r-1}\}$: $\Pi_{\mathcal{C}} = \{0, \ldots, r-1\}$ label set, rank=r.
- $N_{ij}^k = \dim \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_k)$ fusion rules, encodes Grothendieck Ring $K_0(\mathcal{C})$
- $T_{ij} = \delta_{ij}\theta_i$ encodes θ.
- $S_{ij} := Tr_{\mathcal{C}}(c_{ij^*} c_{j^* i})$ encodes non-degeneracy: $\det(S) \neq 0$.

Problem 1

Problem (1)

Algebraically characterize modular categories.

One approach:

Theorem (Davidovich,Hagge,Wang)

A modular category $\mathcal{C}$ is uniquely determine by (an algebraic) quintuple (L, N, F, R, ϵ) called an Modular System.

Remark

- $L \leftrightsquigarrow$labels, $N \leftrightsquigarrow$fusion rules, $F \leftrightsquigarrow$ 6j-symbols, $R \leftrightsquigarrow$braiding, $\epsilon \leftrightsquigarrow$pivotal structure.
- Theoretically useful, rarely available.

Modular Data

Definition

The realizable *Modular Data* of $\mathcal{C}$ is (S, T).

Remark

(S, T) realizable implies $(\sigma(S), \sigma(T))$ realizable for $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$ (using [DHW]).

Problem (1')

Determine a (minimal) set of algebraic/arithmetic conditions $\mathcal{A}$ so that: (S, T) satisfies $\mathcal{A}$ iff realizable.

Remark

In particular we must have:
(S, T) satisfies $\mathcal{A}$ iff $(\sigma(S), \sigma(T))$ satisfy $\mathcal{A}$.

Notation/Facts

Remarks

- For $A \in \mathbb{C}^{r,r}$ define $\mathbb{Q}(A)$ to be the field generated by $\mathbb{Q}$ and the entries of A.
- Set $\mathrm{Gal}(\mathcal{C}) = \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(S))$.
- $\mathrm{FPdim}(X_i) := \max \mathrm{Spec}(N_i)$, $d_i = \dim(X_i) := S_{0i}$
- $\mathfrak{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to S$ and $\mathfrak{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to T$ give a projective representation of $SL(2, \mathbb{Z})$, which can be lifted by $s = S/x$ and $t = T/y$ for some (x, y).

Working Definition

Definition (Bruillard,Ng,R,Wang)

For $S, T \in \mathbb{C}^{(r,r)}$ define $d_j := S_{0j}$, $\theta_j := T_{jj}$, $D^2 := \sum_j d_j^2$, $p_\pm := \sum_j d_j^2 \theta_j^{\pm 1}$. (S, T) is an **admissible modular data** if:

1. $S = S^t$, $d_j \in \mathbb{R}$, $S\overline{S}^t = D^2 Id$, T diagonal, $ord(T) = N < \infty$
2. $(ST)^3 = p_+ S^2$, $p_+ p_- = D^2$, $\frac{p_+}{p_-}$ is a root of 1
3. $N_{ij}^k := \sum_a \frac{S_{ia} S_{ja} \overline{S_{ka}}}{D^2 d_a} \in \mathbb{N}$
4. $\theta_i \theta_j S_{ij} = \sum_k N_{i^* j}^k d_k \theta_k$ where $N_{ii^*}^0$ uniquely defines i^*.
5. $\nu_n(k) := \frac{1}{D^2} \sum_{i,j} N_{ij}^k d_i d_j \left(\frac{\theta_i}{\theta_j}\right)^n$ satisfies: $\nu_2(k) \in \{0, \pm 1\}$
6. $\mathbb{Q}(S) \subset \mathbb{Q}(T)$, $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(S)) \subset \mathfrak{S}_r$, $\mathrm{Aut}_{\mathbb{Q}(S)}(\mathbb{Q}(T)) \cong (\mathbb{Z}_2)^k$.
7. Prime (ideal) divisors of $\langle D^2 \rangle$ and $\langle N \rangle$ coincide in $\mathbb{Z}[\zeta_N]$.

Remarks on Axioms

Remarks

- Realizable implies admissible
- For $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$, (S, T) admissible $\iff (\sigma(S), \sigma(T))$ admissible.
- Unknown: (S, T) determines $\mathcal{C}$ (assuming realizable)?
- $S^2 = D^2 C$ where $C_{ij} = \delta_{i,j^*}$ and $N = \min\{n > 0 : \nu_n(k) = d_k\}$ hold.
- Redundant? Incomplete?

Further definitions

Definition

- $\mathcal{C}$ is **pseudo-unitary** if $D^2 = \sum_j \text{FPdim}(X_j)^2 = \text{FPdim}(\mathcal{C})$.
- $\mathcal{C}$ is **pointed** if $\text{FPdim}(X_i) = 1$ for all i.
- $\mathcal{C}$ is **integral** if $\text{FPdim}(X_i) \in \mathbb{Z}$ for all i.
- $\mathcal{C}$ is **weakly integral** if $\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$.

Example: $\mathcal{C}(\mathfrak{sl}_2, \ell)$

$\mathcal{C}(\mathfrak{sl}_2, \ell)$: modular subquotient of $\text{Rep}(U_q\mathfrak{sl}_2)$ at $q = e^{\pi i/\ell}$.

Example

$\mathcal{C}(\mathfrak{sl}_2, \ell)$ has simple objects $\{X_0 = \mathbf{1}, \ldots, X_{\ell-2}\}$ and:

- $S_{i,j} = \frac{\sin(\frac{(i+1)(j+1)\pi}{\ell})}{\sin(\frac{\pi}{\ell})}$
- $X_1 \otimes X_k = X_{k-1} \oplus X_{k+1}$ where $X_{-1} = X_{\ell-1} = 0$.
- $\theta_j = e^{\pi i (j^2+j)/(4\ell)}$ so $ord(T) = 4\ell$.
- ℓ prime: $\text{Gal}(\mathcal{C}(\mathfrak{sl}_2, \ell)) \cong \langle (0 \cdots \frac{\ell-3}{2})(\frac{\ell-1}{2} \cdots \ell - 2) \rangle \cong \mathbb{Z}_{\frac{\ell-1}{2}}$.
- $\mathcal{C}(\mathfrak{sl}_2, \ell)$ pointed for $\ell = 3$, weakly integral for $\ell = 4, 6$.

Problem 2

Problem (2)

Classify modular categories by rank.

Remark

Weakest: classify by $K_0(\mathcal{C})$, i.e., up to Grothendieck equivalence.
Strongest: classify up to ribbon equivalence.

Theorem (R,Stong,Wang)

Up to Grothendieck equivalence, rank≤ 4 are products or subcategories of $\mathcal{C}(\mathfrak{sl}_2, \ell)$, $\ell \leq 7$.

Generally Feasible?

Theorem (Bruillard,Ng,R,Wang)

There are finitely many modular categories of a given fixed rank r.

Sketch.

- Enough to bound number of **fusion rules** of rank r.
- Enough to bound $\mathrm{FPdim}(\mathcal{C})$ which $= D^2/d_i$ for some i.
- $ord(T) = N$ bounded, so set $\mathcal{S}$ of primes dividing D^2 (in $\mathbb{Z}[\zeta_N]$) **finite**.
- (Evertse): **finitely** many $\mathcal{S}$-unit solutions to $1 + X + \sum_{i=1}^{r-1} x_i = 0$
- (Etingof-Gelaki): $(-D^2, d_1^2, \ldots, d_{r-1}^2)$ is a solution.
- D^2 and all d_i bounded above and below.

□

Low Rank Approach

Fix a rank r.

Remarks

- First classify up to Grothendieck equivalence: determine all possible $K_0(\mathcal{C})$.
- By Verlinde formula enough to determine S-matrices.
- Having S, determine possible T.
- Look for realizations/constructions...
- Step One: Fix abelian subgroup $\text{Gal}(\mathcal{C}) \cong A \subset \mathfrak{S}_r$.
- Step Two: Determine possible $ord(T) = N$
- Step Three: Analyze $\text{SL}(2, \mathbb{Z})$ reps.

Galois Action

Fact

$\phi_k(i) := \frac{S_{ik}}{S_{0k}}$ define (all) characters of $K_0(\mathcal{C})$.

Theorem (de Boer, Gouree)

Let $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$. There exists a unique $\widehat{\sigma} \in \mathfrak{S}_r$ such that

$$\sigma\left(\frac{S_{ik}}{S_{0k}}\right) = \frac{S_{i\widehat{\sigma}(k)}}{S_{0\widehat{\sigma}(k)}}$$

Moreover, $\mathrm{Gal}(\mathcal{C}) \cong \langle \widehat{\sigma} : \sigma \in \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}}) \rangle \subset \mathfrak{S}_r$.

$\exists \epsilon_\sigma : \Pi_{\mathcal{C}} \to \{\pm 1\}$ such that: $S_{ik} = \epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(k) S_{\widehat{\sigma}(i),\widehat{\sigma}^{-1}(k)}$.

Lemma (Bruillard, Ng, R, Wang)

$\mathcal{C}$ integral iff $\mathrm{Gal}(\mathcal{C})(0) = 0$, iff $d_i \in \mathbb{Z}$ for all i.

Galois Action Example

Example

Suppose $r \geq 5$ is odd and $\hat{\sigma} = (0\ 1)(2\ \cdots\ r-1)$ for $\sigma \in \mathrm{Gal}(\mathcal{C})$.

- Applying $S_{ij} = \pm S_{\tau(i)\tau^{-1}(j)}$ yields: $S_{11} = 1$, $\frac{S_{1j}}{d_j} = \epsilon_j$ and $\frac{d_j}{d_2} = \delta_j$ for $\epsilon_j, \delta_j \in \{\pm 1\}$ with $j \geq 2$.
- $Norm_{\mathbb{Q}(S)/\mathbb{Q}}(d_2) = d_2\sigma(d_2) = \pm\frac{d_2^2}{d_1} \in \mathbb{Z}$
- $\frac{S_{1j}}{S_{0j}} = \frac{\epsilon_j}{\delta_j} = \sigma(\frac{S_{1j}}{S_{0j}}) = \frac{S_{1\hat{\sigma}(j)}}{S_{0\hat{\sigma}(j)}}$, so $\frac{\epsilon_j}{\delta_j} = \epsilon$ for all $j \geq 2$.
- Orthogonality of first two rows gives: pause
  $2d_1 + \sum_{j\geq 2} S_{1j}S_{0j} = 2 + \epsilon(r-2)\frac{d_2^2}{d_1}$ so
  $2 = (r-2)\left|\frac{d_2^2}{d_1}\right| \geq (r-2) > 2$, contradiction.

So no such category exists! $r = 3$ gives Ising...

$\mathrm{Aut}_{\mathbb{Q}(S)}(\mathbb{Q}(T)) \cong \mathbb{Z}_2^k$

Definition

Let (S,T) be realizable defining $\rho : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PGL}(r)$ and $(s,t) = (S/x, T/y)$ define a lift $\widehat{\rho} : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(r)$. May choose x so $s^2 = 1$.

Theorem (Dong,Lin,Ng,Schauenburg)

$\mathrm{Aut}_{\mathbb{Q}(S)}(\mathbb{Q}(T)) \cong (\mathbb{Z}_2)^k$, *and* $\mathrm{Aut}_{\mathbb{Q}(s)}(\mathbb{Q}(t)) \cong (\mathbb{Z}_2)^{k'}$.

Fact

Both k and k' are bounded in terms of $\mathbb{Q}(S)$:
s.e.s. $(\mathbb{Z}_2)^k \hookrightarrow (\mathbb{Z}_N)^* \twoheadrightarrow \mathrm{Gal}(\mathcal{C})$ *where* $ord(T) = N$

$\mathrm{Aut}_{\mathbb{Q}(S)}(\mathbb{Q}(T)) \cong \mathbb{Z}_2^k$ Sample results

Lemma

If $|\mathrm{Gal}(\mathcal{C})|$ is odd and $p \mid N$ is prime then $p \equiv 3 \mod 4$.

Lemma

$|\mathrm{Gal}(\mathcal{C})| = p$ prime implies $2p+1$ prime (Sophie Germain primes!)

Proof.

May assume $p > 5$. There exists $q > 3$ prime and $t \geq 1$ s.t. $q^t || N$. $\varphi(q^t) = q^t - q^{t-1} = 2p$. so $t = 1$ and $q = 2p+1$. □

For example, $|\mathrm{Gal}(\mathcal{C})| = 7$ is not possible.

$\mathrm{SL}(2,\mathbb{Z})$ representations

Theorem (Ng,Schauenburg)

- *$ord(T) = N$ (resp. $ord(t) = n$) is minimal such that ρ (resp. $\widehat{\rho}$) factors over $\mathrm{SL}(2,\mathbb{Z}_N)$ (resp. $\mathrm{SL}(2,\mathbb{Z}_n)$). (Congruence Property)*
- *for $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(t))$, $\sigma^2(t_i) = t_{\widehat{\sigma}(i)}$ (Galois Symmetry)*

Theorem (Bruillard,Ng,R,Wang)

If $\rho = \rho_1 \oplus \rho_2$ then $\mathrm{Spec}\,\rho_1(\mathfrak{t}) \cap \mathrm{Spec}\,\rho_2(\mathfrak{t}) \neq \emptyset$.

$SL(2,\mathbb{Z})$ Example

Theorem (Bruillard,Ng,R,Wang)

If $\mathrm{Gal}(\mathcal{C}) \cong \mathbb{Z}_4$ *then* $r \neq 5$.

Proof.

1. $(\mathbb{Z}_2)^k \hookrightarrow (\mathbb{Z}_n)^* \twoheadrightarrow \mathbb{Z}_4$ implies $n \in \{5a, 16a, 32a\}$, $a \mid 24$.
2. $n = 32a$ implies $r \neq 5$.
3. $n \in \{16a, 5a\}$ gives $t = \mathrm{diag}(z, \sigma^2(z), z, \sigma^2(z), w)$, $w^{24} = 1$. (Galois Symmetry)
4. Reducible (repeated eigs.) and no 1-dim'l subreps.
5. subrep. $\widehat{\rho}_1$, $\dim(\widehat{\rho}_1) = 2$ implies $n \neq 16a$.
6. Hence: $t = w\,\mathrm{diag}(\zeta, 1/\zeta) \oplus w\,\mathrm{diag}(\zeta, 1/\zeta, 1)$, $\zeta = e^{2\pi i/5}$. contradicts parity $s^2 = 1$ (inspect 2 and 3-dimensional $SL(2,\mathbb{Z}_5)$ irreps).

Low rank classification

Theorem (Bruillard,Ng,R,Wang and R,Hong)

1. *If* $\mathcal{C}$ *has rank* 5 *then* $\mathcal{C}$ *is Grothendieck equivalent to one of:* $\{\mathcal{C}(\mathfrak{sl}_2, 6), \mathcal{C}(\mathfrak{sl}_2, 11)/\mathbb{Z}_2, \mathcal{C}(\mathfrak{sl}_5, 6), \mathcal{C}(\mathfrak{sl}_3, 7)/\mathbb{Z}_2\}$
2. *If* $\mathcal{C}$ *is integral of rank* ≤ 7 *then* $\mathcal{C}$ *is pointed. If* $\mathcal{C}$ *is weakly integral of rank* 6 *then* $\mathcal{C}$ *is Grothendieck equivalent to* $\mathcal{C}(\mathfrak{sl}_2, 4) \boxtimes \mathcal{C}(\mathfrak{sl}_2, 3)$ *or* $\mathcal{C}(\mathfrak{so}_5, 10)$

Remark

The number of modular categories of rank$\leq r$ is superpolynomial in r. Etingof's example $\mathrm{Rep}(D^\omega \mathbb{Z}_p^m)$.

Conjecture

The number of fusion rings for prime, non-integral modular categories of rank r is polynomially bounded in r.

Problem 3

Definition

Braid group $\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$

Set Up

Let $X \in \mathcal{C}$, $\mathcal{H}_n := \bigoplus_i \mathrm{Hom}(X_i, X^{\otimes n})$ and $\rho^X : \mathcal{B}_n \to GL(\mathcal{H}_n)$ via

$$\sigma_i \to I_X^{\otimes i-1} \otimes c_{X,X} \otimes I_X^{\otimes n-i-1}$$

Problem (3)

Study properties of $(\rho^X, \mathcal{H}_n)$, to distinguish BFCs.

Local $\mathcal{B}_n$ representations: Yang-Baxter eqn.

Definition

(R, V) is a **braided vector space** if $R \in Aut(V \otimes V)$ satisfies

$$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R)$$

Induces a sequence of local $\mathcal{B}_n$-reps $(\rho^R, V^{\otimes n})$ by

$$\rho^R(\sigma_i) = I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}$$

$$v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \xrightarrow{\rho^R(\sigma_i)} v_1 \otimes \cdots \otimes R(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n$$

Localize!

Definition (R,Wang)

A **localization** of a sequence of $\mathcal{B}_n$-reps. (ρ_n, V_n) is a braided vector space (R, W) and injective algebra maps $\tau_n : \mathbb{C}\rho_n(\mathcal{B}_n) \to \mathrm{End}(W^{\otimes n})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & & \\ \downarrow{\scriptstyle \rho_n} & \searrow^{\rho^R} & \\ \mathbb{C}\rho_n(\mathcal{B}_n) & \xrightarrow{\tau_n} & \mathrm{End}(W^{\otimes n}) \end{array}$$

Quasi-localization also defined: see [Galindo,Hong,R. 2013].

Example $\mathcal{C}(\mathfrak{sl}_2, 4)$

Let $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Theorem (Franko,R,Wang 2006)

$(R, \mathbb{C}^2)$ *localizes* $(\rho^X, \mathcal{H}_n)$ *for* $X = X_1 \in \mathcal{C}(\mathfrak{sl}_2, 4)$

Theorem

For $X \in \text{Rep}(DG)$, $\mathcal{B}_n$*-reps.* $(\rho^X, \mathcal{H}_n)$ *always localizable.*

Remark

$\mathcal{B}_n$ acts on both $X^{\otimes n}$ and $\mathcal{H}_n$. The first is local the second is localizable.

Results

Theorem (R,Wang)

For $X \in \mathcal{C}(\mathfrak{sl}_2, \ell)$ $(\rho^X, \mathcal{H}_n)$ *localizable iff* $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$.

Theorem (R,Wang 2012)

Suppose that each $\mathrm{Hom}(X_i, X^{\otimes n})$ *is irreducible as a* $\mathcal{B}_n$*-rep. then Localizable* $\Rightarrow$ *Weakly Integral.*

proof: Spectral Graph Theory

Localization Conjecture

Conjecture (Localization)

$(\rho^X, \mathcal{H}_n)$ is (quasi-)Localizable for all $X \in \mathcal{C}$ if and only if $\mathcal{C}$ is Weakly Integral.

Unitary BVS Conjecture

Question

How do local braid group representations behave?

Conjecture (BVS)

If (R, V) is a unitary, finite order BVS then $|\rho^R(\mathcal{B}_n)| < \infty$.

Reduction

Theorem

Suppose (R, V) is unitary, finite order BVS. Then if $\rho^R(\mathcal{B}_n)$ is finite modulo its center then $|\rho^R(\mathcal{B}_n)| < \infty$

Proof.

- restrict to irred. subrep. $(\rho_i^R, W_i) \subset (\rho^R, V^{\otimes n})$.
- $\rho_i^R(\sigma_j)$ has finite order, so for $z = xId_{W_i} \in Z(\rho_i^R(\mathcal{B}_n))$, $\det(z) = x^{\dim(W_i)}$ is a root of unity.
- Thus z has finite order so $Z(\rho^R(\mathcal{B}_n))$ is f.g. torsion, hence finite.

□

Group Type BVSs

Definition

(R, V) is of **group type** if there exist $g_i \in GL(V)$ and a basis $\{x_i\}$ such that $R(x_i \otimes x_j) = g_i(x_j) \otimes x_i$.

Remark

Group-type BVSs are important to Andruskiewitsch-Schneider classification program for pointed Hopf algebras.

Theorem (Galindo,R)

BVS Conjecture true for group type BVSs.

proof Pass to Yetter-Drinfeld modules over $\mathbb{C}[G]$, enough to show $|G| < \infty$.

Property F

Definition

A braided fusion category has **Property F** if $|\rho^X(\mathcal{B}_n)| < \infty$ for all X.

Theorem (Etingof,R,Witherspoon)

$\text{Rep}(D^\omega G)$ *has property* **F**. *And hence so does any* **group theoretical** *BFC.*

Theorem (Jones,Freedman-Larsen-Wang)

$\mathcal{C}(\mathfrak{sl}_k, \ell)$ *has property* **F** *if and only if weakly integral.*

Locality and Property **F**

Conjecture

Let $\mathcal{C}$ be a braided fusion category. Then the following are equivalent:

1. $\mathcal{C}$ has Property F
2. $(\rho^X, \mathcal{H}_n)$ is quasi-localizable for all X
3. $\mathcal{C}$ is weakly integral.

Mercí!