Classification of Modular Categories

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Definition
A **modular category** $\mathcal{C}$ (over $\mathbb{C}$) is
monoidal: $(\otimes, 1)$,
  
  semisimple: $X \cong \bigoplus_i m_iX_i$,
  
  linear: $\text{Hom}(X, Y) \in \text{Vec}_\mathbb{C}$,
  
  rigid: $X^* \otimes X \mapsto 1 \mapsto X \otimes X^*$,
  
  finite rank: $\text{Irr}(\mathcal{C}) = \{1 = X_0, \ldots, X_{r-1}\}$,
  
  spherical: $u_X\theta_X : X \cong X^{**}$, $\text{dim}(X) \in \mathbb{R}$,
  
  braided: $c_{X,Y} : X \otimes Y \cong Y \otimes X$,
  
  modular: $\det(\text{Tr}_\mathcal{C}(c_{X_i,X_j^*}c_{X_j^*,X_i})) \neq 0$.

Remark
- $u_X$ is Drinfeld isomorphism, $\theta$ is ribbon structure, satisfying,
  e.g.: $\theta_{X \otimes Y} = \theta_X \otimes \theta_Y(c_Y,x c_X,y)$
- $\text{Tr}_\mathcal{C} : \text{End}(X) \rightarrow \mathbb{C}$ is the pivotal trace.
Key Data

- **Fusion rules:** $X_i \otimes X_j \cong \bigoplus_k N_{ij}^k X_k$
- **(Modular) S-matrix:** $S_{ij} := \text{Tr}_C(cX_i, X_j^* cX_j^*, x_i)$
- **(Dehn twist) T-matrix:** $T_{ij} = \delta_{ij}\theta_i$
- **(Congruence subgroup) Level:** $N = \text{ord}(T)$
- **(Quantum) Dimensions:** $d_i := S_{i0}$ and $\text{dim}(C) := \sum_{i=0}^{r-1} d_i^2$
- **Ambient fields:** $\mathbb{Q}(S) := \mathbb{Q}(S_{ij})$ and $\mathbb{Q}(T) := \mathbb{Q}(\theta_i)$.
- **(Dedekind domain) Ring of integers:** $\mathcal{O}_{\mathbb{Q}(T)} = \mathbb{Z}[\zeta_N]$. 
Definition (In Coordinates)

For \( S, T \in \mathbb{C}^{(r,r)} \) define \( d_j := S_{0j}, \theta_j := T_{jj}, D^2 := \sum_j d_j^2, p_\pm := \sum_j d_j^2 \theta_j^{\pm 1} \). \((S, T)\) admissible if:

1. \( S = S^t, SS^t = D^2 I_d, T \) diagonal, \( \text{ord}(T) = N < \infty \)
2. \( (ST)^3 = p_+ S^2, p_+ p_- = D^2, \left( \frac{p_+}{p_-} \right)^N = 1 \)
3. \( N^k_{ij} := \sum_a \frac{S_{ia} S_{ja} S_{ka}}{D^2 d_a} \in \mathbb{N} \)
4. \( \theta_i \theta_j S_{ij} = \sum_a N^k_{i*j} d_k \theta_k \) where \( N^0_{ii*} \) uniquely defines \( i^* \).
5. \( \nu_n(k) := \frac{1}{D^2} \sum_{i,j} N^k_{ij} d_i d_j \left( \frac{\theta_i}{\theta_j} \right)^n \in \mathbb{Z}[\zeta_N] \) satisfies:
   \( \nu_2(k) \in \{0, \pm 1\} \)
6. \( \mathbb{Q}(S) \subset \mathbb{Q}(T), \text{Aut}_{\mathbb{Q}} \mathbb{Q}(S) \subset \mathfrak{S}_r, \text{Aut}_{\mathbb{Q}(S)} \mathbb{Q}(T) \cong (\mathbb{Z}_2)^k \).
7. Prime (ideai) divisors of \( \langle D^2 \rangle \) and \( \langle N \rangle \) coincide in \( \mathbb{Z}[\zeta_N] \).

Conjecture: Any admissible \((S, T)\) determines a modular category.
Pointed: $C(A, q)$, $A$ finite abelian group, $q$ non-degenerate quad. form on $A$.

Group-theoretical: $\mathcal{D} \subset \text{Rep}(D^\omega G)$, $\omega$ a 3-cocycle on $G$ a finite group.

Quantum groups/Kac-Moody algebras: subquotients of $\text{Rep}(U_q g)$ at $q = e^{\pi i/\ell}$ or level $k$ integrable $\hat{g}$-modules. e.g.
- $SU(N)_k = C(\mathfrak{sl}_N, N + k)$,
- $SO(N)_k$,
- $Sp(N)_k$,
- for $\gcd(N, k) = 1$, $\text{PSU}(N)_k \subset SU(N)_k$ “even half”

Drinfeld center: $Z(\mathcal{D})$ for spherical fusion category $\mathcal{D}$. 
Example

\(SU(2)_{\ell-2}\) from \(\text{Rep}(U_q \mathfrak{sl}_2)\) at \(q = e^{\pi i/\ell}\)

- Simple objects \(\{X_0 = 1, X_1, \ldots, X_{\ell-2}\}\)

- \(S_{ij} = \frac{\sin\left(\frac{(i+1)(j+1)\pi}{\ell}\right)}{\sin\left(\frac{\pi}{\ell}\right)}\)

- \(X_1 \otimes X_k \cong X_{k-1} \oplus X_{k+1}\) for \(1 \leq k \leq \ell - 3\)

- \(\theta_j = e^{\frac{\pi i(j^2+2j)}{2\ell}}\) so \(\text{ord}(T) = 4\ell\)

- \(D^2 = \dim SU(2)_{\ell-2} = \sum_{n=1}^{\ell-1} [n]^2\) where \([n] = \frac{q^n - q^{-n}}{q - q^{-1}}\).
Tambara Yamagami Categories

Definition

$\mathcal{C}$ is *weakly integral* if $\dim(\mathcal{C}) \in \mathbb{N}$.

Example

Let $A$ be a finite abelian group, $\chi$ a non-degenerate bicharacter on $A$ and $\nu$ a sign. Tambara and Yamagami defined a spherical fusion category $TY(A, \chi, \nu)$ with simple objects $A \cup \{m\}$ with:

- $m \otimes m = \sum_{a \in A} a$
- $m \otimes a = m$
- $a \otimes b = ab$

$\dim(TY(A, \chi, \nu)) = 2|A|$. $\mathcal{Z}TY(A, \chi, \nu)$ is modular.
For the rest of this talk, we assume \( \dim(X) > 0 \) for all \( X \).

Problem

Classify “small” modular categories.

E.g.:

- Bounded rank: \( r < M \),
- few primes: \( |\{p : p \mid \dim(C)\}| < M \). For \( p \in \mathbb{Z} \), stay for Naidu’s talk
- Bounded level: \( \text{ord}(T) < M \)
Theorem (Bruillard, Ng, R, Wang 2013)

There are finitely many modular categories of a given rank $r$.

History:

- (2005) Verified for: fusion categories with $\text{dim}(C) \in \mathbb{N}$ (Etingof, Nikshych and Ostrik), rank $= 3$ (Ostrik).
Warm Up

Theorem (E. Landau 1903)
For any \( r \in \mathbb{N} \), there are finitely many groups \( G \) with \( |\text{Irr}(G)| = r \).

Proof.
Use class equation:
\[
|G| = \sum_{i=1}^{r} |\overline{g_i}|,
\]
where \( \overline{g_i} \) distinct conjugacy classes. Set \( x_i = [G : C(g_i)] \) (index of centralizers) to get
\[
1 = \sum_{i=1}^{r} \frac{1}{x_i}.
\]

\( x_i \leq a(r) \) where \( a(1) = 2, a(2) = 3, a(n) = a(n-1)a(n-2) + 1 \) is Sylvester’s sequence. Therefore \( |G| = \max_i x_i \) is bounded. So finitely many multiplication tables. \( \square \)
Key Steps

Proof.

1. By [R, Stong, Wang ’08] enough to bound $\dim(C)$.
2. By [Evertse ’84] enough to bound
   
   $$\bigcup_{\text{rank}(C)=r} \{ p \in \text{Spec } \mathbb{Z}[\zeta_N] : p|\langle \dim(C) \rangle \}$$

3. By Cauchy Theorem [Bruillard, Ng, R., Wang] enough to bound $M_r := \max\{ N = \text{ord}(T) : \text{rank}(C) = r \}$.
Some Details

- Congruence Subgroup Property: gives bound on $M_r$
- Cauchy: prime divisors of $\langle \dim(C) \rangle$ and $\langle N \rangle$ in $\mathbb{Z}[\zeta_N]$ coincide.
- Evertse: finite number of non-degenerate solutions to $0 = 1 + x_0 + \cdots + x_{r-1}$ where $x_i$ are $S$-units for any finite set of primes $S$. Notice: $0 = 1 - \dim(C) + d_1 + \cdots + d_{r-1}$, so $\dim(C)$ bounded!

- Example ($C = \text{SU}(2)_3$)
  $N = \text{ord}(T) = 20$ and $\dim(C) = 5 + \sqrt{5}$. In $\mathbb{Z}[\zeta_{20}]$,
  - $\langle 5 + \sqrt{5} \rangle = \langle 2 \rangle \langle 1 - (\zeta_{20})^4 \rangle^2$
  - $\langle 20 \rangle = \langle 2 \rangle^2 \langle 1 - (\zeta_{20})^4 \rangle^4$
Theorem (R, Hong, Stong, Bruillard, Ng, Wang, Ostrik)

If $2 \leq \text{rank}(C) \leq 5$ then $C$ has the same fusion rules as one of:

- $\text{PSU}(2)_3$ (Fibonacci), $\text{SU}(2)_1$ (pointed)
- $\text{PSU}(2)_5$, $\text{SU}(2)_2$ (Ising), $\text{SU}(3)_1$ (pointed)
- $\text{PSU}(2)_7$, $\text{SU}(2)_3$, $\text{SU}(4)_1$, products.
- $\text{SU}(2)_4$, $\text{PSU}(2)_9$, $\text{SU}(5)_1$, $\text{PSU}(3)_4$. 
Theorem (R, Bruillard)

- Suppose $\mathcal{C}$ has $\text{ord}(T) \in \{2, 3, 4\}$. Then $\mathcal{C} \subset \text{Rep}(D^\omega G)$ for some 2- or 3-group $G$.
- Suppose $\dim(\mathcal{C})$ is odd and $\text{rank}(\mathcal{C}) \leq 11$. Then $\mathcal{C}$ is pointed.

Remark

- $\text{ord}(T) = 6$ implies integral, but not $\mathcal{C} \subset \text{Rep}(D^\omega G)$.
- Open: $\dim(\mathcal{C})$ odd, $13 \leq \text{rank}(\mathcal{C}) \leq 23$ implies pointed?
- Open: $\dim(\mathcal{C})$ odd implies group-theoretical?
Theorem (R, Ng, Bruillard, Wang, Galindo, Plavnik)

If $C$ is:

- weakly integral with $\text{rank}(C) \leq 7$ or
- $\dim(C) = 4m$ with $m$ odd and square-free

then $C \cong D \boxtimes F$ where $D, F \subset \mathcal{Z}TY(A, \chi, \nu)$.

More explicitly: $D, F$ are pointed, Ising or $TY(\mathbb{Z}_{2k+1}, \chi, \nu)^{\mathbb{Z}_2}$.

Lemma

Suppose $C$ is weakly integral modular category with exactly one simple (class) $X$ with $\dim(X) \not\in \mathbb{Z}$. Then $C$ is an Ising category.

Problem

$\dim(X) \in \mathbb{Z}$ for all $X$ implies $C \equiv \text{Rep}(H)$, $H$ quasi-Hopf. What about $\dim(C) \in \mathbb{Z}$?
Thank you!

Based on arXiv: 1310.7050 and 1411.2313