Localizing Braided Fusion Categories

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July 2011
based on arXiv:1009.0241 (to appear CMP)
and arXiv:1105.5048

Outline

1. Sequences of $\mathcal{B}_n$-representations and Localizability
   - Sequences
   - Localization
   - Examples
2. Speculations and Further Directions
   - Preliminary Results and Conjectures
   - Work with Galindo and Hong
3. Motivation: Quantum Computation
   - Quantum Circuit Model
   - Topological Model

The Braid Group

A key role is played by the braid group:

Definition

$\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$

General Context

Let $\iota : \mathcal{B}_n \to \mathcal{B}_{n+1}$, $\iota(\sigma_i) = \sigma_i$ for $i \leq n-1$.

Definition

A sequence of braid representations is a family of $\mathcal{B}_n$-reps (ρ_n, V_n) and *injective* algebra maps τ_n such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & \xrightarrow{\rho_n} & \mathbb{C}\rho_n(\mathcal{B}_n) \\ \downarrow \iota & & \downarrow \tau_n \\ \mathbb{C}\mathcal{B}_{n+1} & \xrightarrow{\rho_{n+1}} & \mathbb{C}\rho_{n+1}(\mathcal{B}_{n+1}) \end{array}$$

Braided Vector Spaces

Definition

(R, V) is a **braided vector space** if $R \in Aut(V \otimes V)$ satisfies
$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R)$

Induces a sequence of $\mathcal{B}_n$-reps $(\rho^R, V^{\otimes n})$ by

$$\rho^R(\sigma_i) = I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}$$

Braided Fusion Categories

Categorical construction:

- Fix $X \in \mathcal{C}$ (strict) braided fusion category
- Braiding isomorphism $c_{X,X} \in \mathrm{End}(X^{\otimes 2})$ induces: $\psi_n : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$ via $\sigma_i \to I_X^{\otimes i-1} \otimes c_{X,X} \otimes I_X^{\otimes n-i-1}$
- $\mathbb{C}\mathcal{B}_n$ acts via ψ_n on the $\mathrm{End}(X^{\otimes n})$-module

$$W_n^X := \bigoplus_{Y \text{simple}} \mathrm{Hom}(Y, X^{\otimes n})$$

- Denote (ρ_X, W_n^X).

Examples from Quantum Groups

Example

The (semisimple) subquotients $\mathcal{C}(\mathfrak{g}, \ell)$ of $\text{Rep}(U_q\mathfrak{g})$ for $\mathfrak{g}$ a Lie algebra and $q = exp(\pi i/\ell)$ are braided fusion categories.
E.g. $\mathfrak{g} = \mathfrak{sl}_2$ with X the "vector representation" corresp. to Jones representations of $\mathcal{B}_n$.

Notation

Denote by $\rho^{(\ell)}$ the $\mathcal{B}_n$-rep. associated with $X \in \mathcal{C}(\mathfrak{sl}_2, \ell)$.

Question: Square Peg, Round Hole?

Notice that $(\rho^R, V^{\otimes n})$ is local: $\rho^R(\sigma_i)$ acts non-trivially only on adjacent tensor factors:

$$v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \xrightarrow{\rho^R(\sigma_i)} v_1 \otimes \cdots \otimes R(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n$$

Question

Given a sequence (ρ_n, V_n), when can it be realized via braided v.s. (R, V)? "Localized"

Formal Definition

Definition

A localization of a sequence of $\mathcal{B}_n$-reps. (ρ_n, V_n) is a braided vector space (R, W) such that for all $n \geq 2$:There exist injective algebra maps $\varphi_n : \mathbb{C}\rho_n(\mathcal{B}_n) \to \mathrm{End}(W^{\otimes n})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & & \\ \downarrow{\scriptstyle \rho_n} & \searrow^{\rho^R} & \\ \mathbb{C}\rho_n(\mathcal{B}_n) & \xrightarrow{\varphi_n} & \mathrm{End}(W^{\otimes n}) \end{array}$$

Combinatorially...

If (R, W) localizes (ρ_n, V_n),

- Decompose (ρ_n, V_n): $V_n \cong \bigoplus_{i \in J_n} V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- then $W^{\otimes n} \cong \bigoplus_{i \in J_n} \mu_n^i V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- with $\mu_n^i > 0$ (multiplicities)

Remarks

- $\dim(V_n) \neq d^n$ (usually), so extra copies of some $V_n^{(i)}$ needed.
- (R, W) uniformly localizes for all n.
- $\overrightarrow{\mu}_n$ localization vector.

Obvious Examples: q.t. Hopf algebras

Theorem

Let $X \in \mathrm{Rep}(H)$, for (H, R) a f.d. s.s. quasi-triangular Hopf algebra. Then (ρ_X, W_n^X) is localizable with localization $(R|_{X^{\otimes 2}}, X)$.

Proof.

Double-commutant argument: $X^{\otimes n} \cong \bigoplus_Y \mathrm{Hom}(Y, X^{\otimes n}) \otimes Y$. □

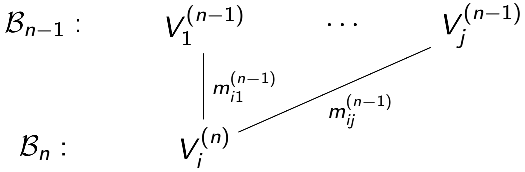
Bratteli Diagrams

Consider irreducible $\mathcal{B}_n$-rep $V_i^{(n)}$.
How does $V_i^{(n)}|_{\mathcal{B}_{n-1}}$ decompose?

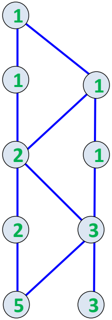
$$V_i^{(n)} \cong \bigoplus_j m_{ij}^{(n-1)} V_j^{(n-1)}$$

Recorded in **Inclusion Matrix** $G^{(n-1)} := [m_{ij}^{(n-1)}]_{ij}$ or

Bratteli diagram

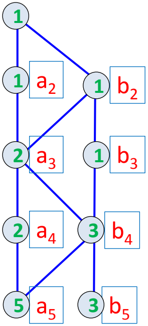


Example: $\mathcal{C}(\mathfrak{sl}_2, 5)$



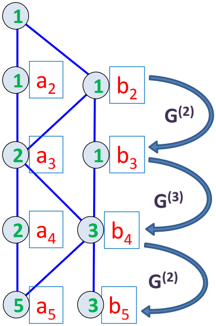
If (R, V) localizes ρ^5

Example: $\mathcal{C}(\mathfrak{sl}_2, 5)$



If (R, V) localizes ρ^5
with mult. vectors (a_n, b_n)

Example: $\mathcal{C}(\mathfrak{sl}_2, 5)$



If (R, V) localizes ρ^5 with mult. vectors (a_n, b_n) then by Perron-Frobenius Theorem

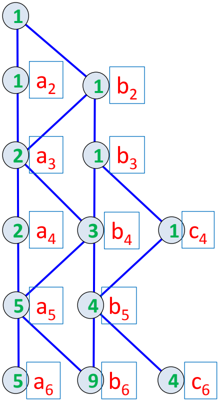
$$G^{(3)}G^{(2)}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \lambda \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

where $G^{(3)}G^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\lambda = \left(\frac{1+\sqrt{5}}{2}\right)^2,\ a_2, b_2 \in \mathbb{Z}.$$

Impossible!

Example: $\mathcal{C}(\mathfrak{sl}_2, 6)$



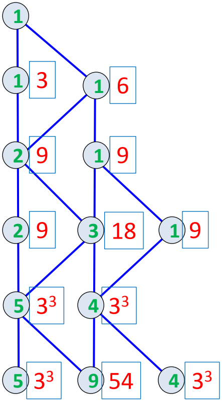
If (R, V) localizes ρ^6 with $\dim(V) = k$ then

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_4 \\ b_4 \\ c_4 \end{pmatrix} = \lambda \begin{pmatrix} a_4 \\ b_4 \\ c_4 \end{pmatrix}$$

and $2a_4 + 3b_4 + c_4 = k^4$

$k = \lambda = 3,\ a_4 = b_4/2 = c_4 = 9$ works!

Example: $\mathcal{C}(\mathfrak{sl}_2, 6)$



Is there a 9×9 R-matrix?

$$\gamma \begin{pmatrix} \omega & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega \\ 0 & \omega & 0 & 0 & 0 & \omega & 1 & 0 & 0 \\ 0 & 0 & \omega & \omega^2 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 & \omega & 0 & 0 & 0 & \omega^2 & 0 \\ \omega & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \omega & \omega & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 1 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & \omega^2 & 0 & 0 & 0 & \omega & 0 \\ 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & \omega \end{pmatrix}$$

Localizes ρ^6.

First Results

Theorem (R,Wang)

$\mathcal{B}_n$ reps ρ^ℓ localizable if, and only if $\ell \in \{2,3,4,6\}$

Note: $\mathrm{FPdim}(X) \in \{1, \sqrt{2}, \sqrt{3}\}$

Theorem (R,Wang)

If $\psi_n : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$ is surjective and (ρ_X, W_n^X) is localizable then $\mathrm{FPdim}(X)^2 \in \mathbb{N}$.

Localization Conjecture

Conjecture (R,Wang)

For unitary (ρ_X, W_n^X) TFAE:

(L) ρ_X is localizable, with R finite order

(F) $|\rho_X(B_n)| < \infty$

(W) $\text{FPdim}(X)^2 \in \mathbb{N}$

Braided Vector Space Conjecture

Conjecture (R,Wang)

Suppose (R, V) is a braided v.s. with:

- R Unitary
- R finite order ($R^k = I$)

Then $\rho^R(\mathcal{B}_n)$ is finite for all n.

Further Directions

With Galindo and Hong:

1. realization free (categorical) version defined.
2. quasi- and generalized localizations studied.
3. Unitarity issues dealt with using Galindo's Clifford Theory.
4. quasi-localizations are local up to conjugation, so $V \in \text{Rep}(H)$ for a quasi-triangluar quasi-Hopf H leads to quasi-localizations.

Generalized Y-B equation

Definition

Fix $k > m$ integers, V a vector space. The generalized Yang-Baxter equation is:

$$(R \otimes I_V^{\otimes m})(I_V^{\otimes m} \otimes R)(R \otimes I_V^{\otimes m}) = (I_V^{\otimes m} \otimes R)(R \otimes I_V^{\otimes m})(I_V^{\otimes m} \otimes R)$$

where $R \in \mathrm{End}(V^{\otimes k})$.

- R is a (k, m)-gYB operator if it also satisfies *far commutivity*, i.e. braid relation (R2).
- corresponds to generalized localizations.

QCM state space

Fix d$\in \mathbb{Z}$

Definition

Let $V = \mathbb{C}^d$. The n-qudit **state space** is the n-fold tensor product:

$$\mathcal{H}(n) := V \otimes V \otimes \cdots \otimes V.$$

Gates and Circuits

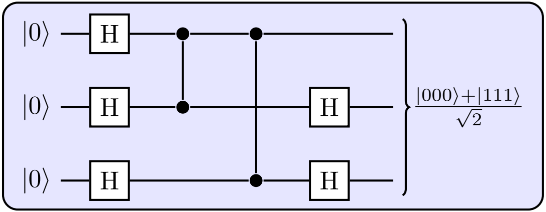
A **quantum gate** is a unitary operator $U_i \in \mathbf{U}(\mathcal{H}(n_i))$
A **gate set** $S = \{U_i\}$ is a collection of gates.

Definition

A **quantum circuit** for $U \in \mathbf{U}(\mathcal{H}(n))$ on S is:

- $G_1, \ldots, G_m \in \mathbf{U}(\mathcal{H}(n))$
- where $G_i = I_V^{\otimes a} \otimes U_j \otimes I_V^{\otimes b}$, $U_j \in S$ and
- $G_1 \cdot G_2 \cdots G_m = U$

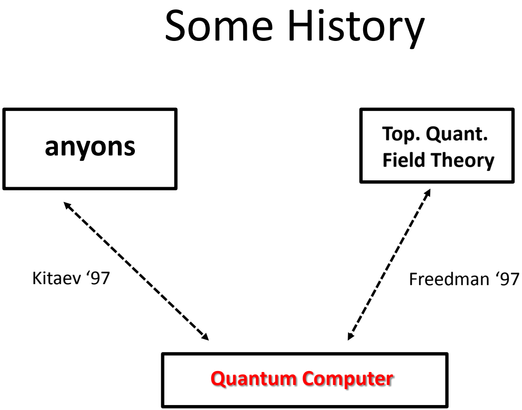
Schematic of a Quantum Circuit



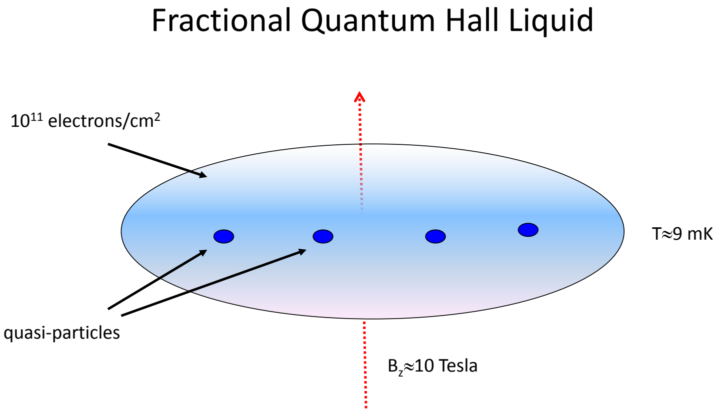
Remark

Here input is $|000\rangle = |0\rangle^{\otimes 3} \in (\mathbb{C}^2)^{\otimes 3}$ and $H \in \mathbf{U}(2)$.
vertical bars are other gates (controlled-phase).

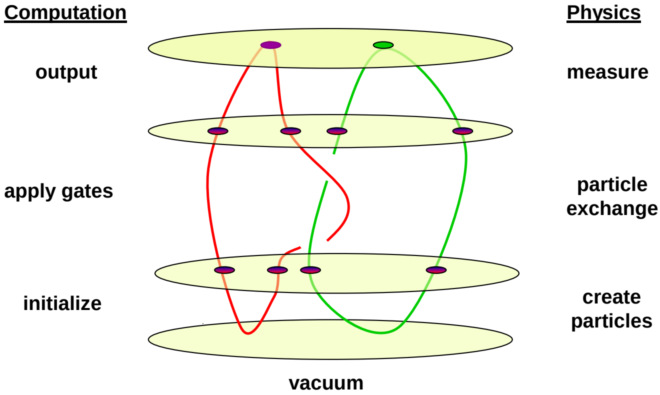
Origins of Topological Model



Example: FQH Liquid Cartoon



Topological Model (non-adaptive)

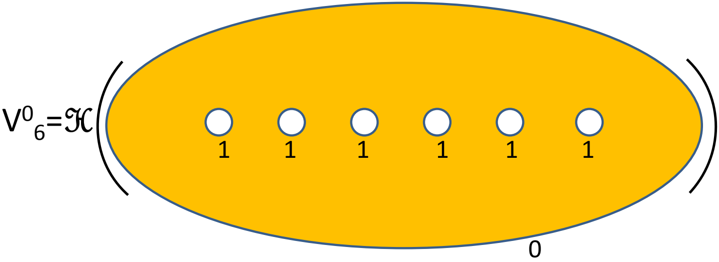


Example: Fibonacci Theory

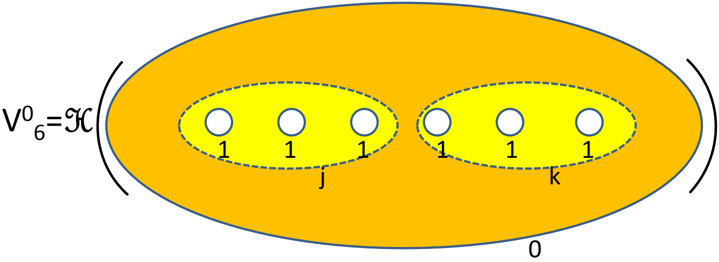
Input: modular category $\mathcal{C}(\mathfrak{g}_2, 15)$:

- $\mathcal{L} = \{0, 1\}$
- Define: $V_k^i := \mathcal{H}(D^2 \setminus \{z_i\}_{i=1}^k; i, 1, \cdots, 1)$
- $\dim V_n^i = \begin{cases} Fib(n-2) & i = 0 \\ Fib(n-1) & i = 1 \end{cases}$

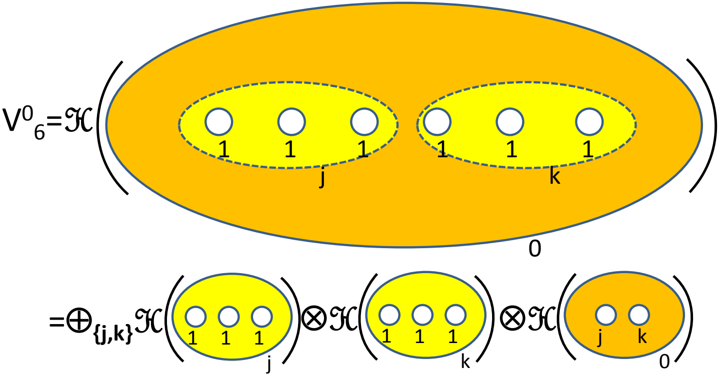
Example: V_6^0



Example: V_6^0



Example: V_6^0



$$= \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C} \oplus \mathbf{C} \otimes \mathbf{C} \otimes \mathbf{C} = \mathbf{C}^4 \oplus \mathbf{C}$$

Motivating Question

Question

When can a given topological quantum computation model be exactly and efficiently simulated by a quantum circuit model?

Partial Answer

If Localization Conjecture holds, only when NO quantum speedup is achieved (non-universal models).

Ask me later if you are interested in this angle....

Thank You!