A Finiteness Property for Braided Fusion Categories

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Outline

1. The Conjecture
   - Braided Fusion Categories
   - Dimensions and Braid Representations

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   - Braided Fusion Categories
   - Dimensions and Braid Representations
2. Empirical Evidence
   - Quantum Groups
   - Group Theoretical Categories

Outline

Some Axioms

Definition

A fusion category $\mathcal{C}$ is a monoidal category that is:

- $\mathbb{C}$-linear, abelian
- finite rank: simple classes $\{X_0 := \mathbf{1}, X_1, \ldots, X_{m-1}\}$
- semisimple
- rigid: duals X^*, $b_X : \mathbf{1} \to X \otimes X^*$, $d_X : X^* \otimes X \to \mathbf{1}$
- compatibility...

Braiding

Definition

A braided fusion (BF) category has (a natural family of) isomorphisms:

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying, e.g.

$$c_{X,Y\otimes Z} = (\mathrm{Id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{Id}_Z)$$

Further structure:

- ribbon fusion categories: braiding and $*$ compatible
- modular categories: Müger center trivial.

Some (familiar) Sources of Braided Fusion Categories

Example

Quantum group $U = U_q\mathfrak{g}$ with $q^\ell = -1$.

- subcategory of *tilting modules* $\mathcal{T} \subset \text{Rep}(U)$
- quotient $\mathcal{C}(\mathfrak{g}, \ell)$ of $\mathcal{T}$ by *negligible morphisms* is a BF category (ribbon).

Example

G a finite group, ω a 3-cocycle

- semisimple quasi-triangular quasi-Hopf algebra $D^\omega G$
- $\text{Rep}(D^\omega G)$ is a BF category (modular).

Generally, Drinfeld center $\mathcal{Z}(\mathcal{C})$ is BF if $\mathcal{C}$ is a fusion category.

Grothendieck Semiring

Definition

$Gr(\mathcal{C}) := (Obj(\mathcal{C}), \oplus, \otimes, \mathbf{1})$ a unital based ring.

- Define matrices $(N_i)_{k,j} := \dim \mathrm{Hom}(X_i \otimes X_j, X_k)$
- Rep. $\varphi : Gr(\mathcal{C}) \to \mathrm{End}(\mathbb{Z}^m)$

$$\varphi(X_i) = N_i$$

- Respects duals: $\varphi(X^*) = \varphi(X)^T$ (self-dual $\Rightarrow$ symmetric)
- If $\mathcal{C}$ is braided, $Gr(\mathcal{C})$ is commutative

Frobenius-Perron Dimensions

Definition

- FPdim(X) is the largest eigenvalue of $\varphi(X)$
- $\text{FPdim}(\mathcal{C}) := \sum_{i=0}^{m-1} \text{FPdim}(X_i)^2$

(a) $\text{FPdim}(X) > 0$

(b) $\text{FPdim} : Gr(\mathcal{C}) \to \mathbb{C}$ is a unital homomorphism

(c) FPdim is unique with (a) and (b).

(Weak) Integrality

Definition

$\mathcal{C}$ is

- integral if $\mathrm{FPdim}(X) \in \mathbb{Z}$ for all X
- weakly integral if $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$

[Etingof,Nikshych,Ostrik '05]: $\mathcal{C}$ integral iff $\mathcal{C} \cong \mathrm{Rep}(H)$, H f.d. s.s. quasi-Hopf alg.

A Consequence

Lemma

$\mathcal{C}$ weakly integral iff $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ for all simple X.

Proof.

Exercise. Use Galois argument. □

The Braid Group

Definition

$\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1$$

Braid Group Representations

Fact

Braiding on $\mathcal{C}$ induces:

$$\Psi_X : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$$

$$\sigma_i \to \mathrm{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{Id}_X^{\otimes n-i-1}$$

- X is *not* always a vector space
- $\mathrm{End}(X^{\otimes n})$ semisimple algebra (multi-matrix).
- simple $\mathrm{End}(X^{\otimes n})$-mods $V_k = \mathrm{Hom}(X^{\otimes n}, X_k)$ become $\mathcal{B}_n$ reps.
- V_k irred. as $\mathcal{B}_n$ reps. if Ψ_X is surjective.

Braid Group Images

Question

Given X and n, what is $\Psi_X(\mathcal{B}_n)$?

(F) Is it finite or infinite?

(U) If unitary and infinite, what is $\overline{\Psi_X(\mathcal{B}_n)}$?

see [Freedman,Larsen,Wang '02], [Larsen,R,Wang '05]

(M) If finite, what are minimal quotients?

see [Larsen,R. '08 AGT]

For example:

- (U): typically $\overline{\Psi_X(\mathcal{B}_n)} \supset \prod_k \mathrm{SU}(V_k)$, V_k irred. subreps.
- (M): $n \geq 5$ solvable $\Psi_X(\mathcal{B}_n)$ implies abelian.

Property F

Definition

Say $\mathcal{C}$ has property **F** if $|\Psi_X(\mathcal{B}_n)| < \infty$ for all X and n.

- $\cdots \subset \Psi_X(\mathcal{B}_n) \subset \Psi_X(\mathcal{B}_{n+1}) \subset \cdots$
  so if no property **F**, $|\Psi_X(\mathcal{B}_n)| = \infty$ for all $n >> 0$
- If $Y \subset X^{\otimes k}$ then $\Psi_X(\mathcal{B}_{kn}) \twoheadrightarrow \Psi_Y(\mathcal{B}_n)$
  so to verify prop. **F**, check for generating X.

First Examples

Examples

	$\mathcal{C}(\mathfrak{sl}_2, 4)$	$\mathcal{C}(\mathfrak{g}_2, 15)$	$\mathrm{Rep}(DS_3)$	$\mathcal{Z}(\frac{1}{2}E_6)$
rank	3	2	8	10
$\mathrm{FPdim}(X_i)$	$\sqrt{2}$	$\frac{1+\sqrt{5}}{2}$	$2, 3$	$\sqrt{3} + \{1, 2, 3\}$
Prop. **F**?	Yes	No	Yes	No

$\frac{1}{2}E_6$ is a non-braided rank 3 fusion category with $X^{\otimes 2} = \mathbf{1} \oplus 2X \oplus Y$, $Y^{\otimes 2} = \mathbf{1}$.

Property **F** Conjecture

Conjecture

A braided fusion category $\mathcal{C}$ has property **F** if and only if it is weakly integral ($\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$).

- Clear for pointed categories ($\text{FPdim}(X_i) = 1$)
- E.g.: does $\text{Rep}(H)$ have prop. **F** for H f.d., s.s., quasi-$\triangle$, quasi-Hopf alg.?

Lie Types A and C

Proposition (Jones '86, Freedman,Larsen,Wang '02)

$\mathcal{C}(\mathfrak{sl}_k, \ell)$ *has property* **F** *if and only if* $\ell \in \{k, k+1, 4, 6\}$.

Proposition (Jones '89, Larsen,R,Wang '05)

$\mathcal{C}(\mathfrak{sp}_{2k}, \ell)$ *has property* **F** *if and only if* $\ell = 10$ *and* $k = 2$.

Approach:

- Take V generating "vector rep." and $q = e^{\pi i/\ell}$
- $\Psi_V(\mathbb{C}\mathcal{B}_n)$ is quotient of $\mathrm{Hecke}_n(q^2)$ or $\mathrm{BMW}_n(-q^{2k+1}, q)$
- only weakly integral in these cases

$$(\mathrm{FPdim}(V) \in \{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, 3\}).$$

Lie types B and D

Conjecture

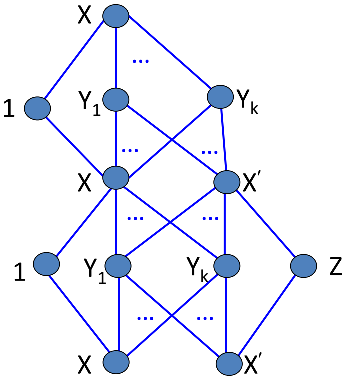
- $\mathcal{C}(\mathfrak{so}_{2k+1}, 4k+2)$ has property **F**
- $\mathcal{C}(\mathfrak{so}_{2m}, 2m)$ has property **F**

- Difficulty: spin objects V_ϵ. Description of $\Psi_{V_\epsilon}(\mathbb{C}\mathcal{B}_n)$?
- $\mathrm{FPdim}(V_\epsilon) \in \{\sqrt{2k+1}, \sqrt{m}\}$
- Verified for $k \leq 4$, $m \leq 5$
- Property **F** fails otherwise [Larsen,R,Wang '05].

Some Details

$\mathcal{P} = \mathcal{C}(\mathfrak{so}_p, 2p)$, p prime
set $X := V_\epsilon$
simples:
$\{\mathbf{1}, Z, X, X', Y_1, \ldots, Y_k\}$
$\mathrm{FPdim}(X) = \mathrm{FPdim}(X') = \sqrt{p}$
$\mathrm{FPdim}(Y_i) = 2$, $\mathrm{FPdim}(Z) = 1$
$\dim \mathrm{Hom}(X^{\otimes n}, X) = \frac{p^{\frac{n-1}{2}}+1}{2}$
$\dim \mathrm{Hom}(X^{\otimes n}, X') = \frac{p^{\frac{n-1}{2}}-1}{2}$

Bratteli Diagram



Guesses?

Look for a series of finite (simple) groups with irreps of dimensions:

$$\frac{p^{\frac{n-1}{2}}+1}{2} \text{ and } \frac{p^{\frac{n-1}{2}}-1}{2}$$

Any guesses?

Conjecture

PSp($2n, p$) (Weil representation.)

This has been verified for $p = 3, 5$ and 7

Exceptional Type Example

Proposition

Property **F** *conjecture is true for* $\mathcal{C}(\mathfrak{g}_2, \ell)$.

Proof.

(outline) Let X be "7-dimensional" object, assume $3 \mid \ell$.

1. For $\ell >> 0$, $\dim \mathrm{Hom}(X^3, X) = 4$ and $\mathcal{B}_3$ acts irreducibly.
2. $Spec(\Psi_X(\sigma_1))$: $\{q^{-12}, q^2, -q^{-6}, -1\}$.
3. $|\Psi_X(\mathcal{B}_3)| = \infty$ for $0 << \ell$ (use [R,Tuba '09?])
4. Check $\mathrm{FPdim}(X)^2 \notin \mathbb{Z}$. Verify for small ℓ.

□

For $3 \nmid \ell$, use [R '08] for FPdim.

Main Tool

$\mathcal{C}$ is group-theoretical if

- $\mathcal{Z}(\mathcal{C}) \cong \text{Rep}(D^\omega G)$ [Natale '03], or
- $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{P})$, $\mathcal{P}$ a *pointed* category.

Proposition (Etingof,R.,Witherspoon '08)

Braided group-theoretical categories $\mathcal{C}$ have property **F**.

Proof.

Braided functor $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C}) \cong \text{Rep}(D^\omega G)$.
Reduces to $\text{Rep}(D^\omega G)$.
$\mathcal{B}_n$ acts on $(D^\omega G)^{\otimes n}$ as monomial group. □

Useful Criterion

Proposition (Drinfeld,Gelaki,Nikshych,Ostrik)

An integral modular category $\mathcal{C}$ is group-theoretical if and only if there exists a $\mathcal{D} \subset \mathcal{C}$ such that

- *$\mathcal{D}$ is symmetric and*
- *$(\mathcal{D}')_{ad} \subset \mathcal{D}$*

- Here $\mathcal{D}'$ is the Müger center:

$$\{X : c_{X,Y} c_{Y,X} = \mathrm{Id}_{X \otimes Y} \, \mathrm{all} \, Y \in \mathcal{D}\}$$

- $\mathcal{L}_{ad}$ is "spanned" by subobjects of all $X \otimes X^*$.

Some Applications

Results (Naidu,R)

- If $\sqrt{2k+1} \in \mathbb{Z}$, $\mathcal{C}(\mathfrak{so}_{2k+1}, 4k+2)$ has property **F**.
- If $\sqrt{m} \in \mathbb{Z}$, $\mathcal{C}(\mathfrak{so}_{2m}, 2m)$ has property **F**.
- If $\mathcal{C}$ a BF category with $\mathrm{FPdim}(X_i) \in \{1,2\}$ and $X^* \cong X$ for all X, $\mathcal{C}$ has property **F**.
- If $\mathcal{C}$ is an integral modular category with $\mathrm{FPdim}(\mathcal{C}) < 36$, then $\mathcal{C}$ has property **F**. cf. [Natale '09?]

Approach: show certain subcategories are group-theoretical.

More Examples

Example

Let A be an abelian group, χ nondeg. sym. bilinear form on A and $\tau = \pm 1/\sqrt{|A|}$.
Tambara-Yamagami categories $\mathcal{TY}(A, \chi, \tau)$ have simple objects $A \cup \{m\}$
with fusion rules:

$$m \otimes a = m, \quad m^{\otimes 2} = \sum_{a \in A} a$$

and associativity defined via χ.
$\mathcal{TY}(A, \chi, \tau)$ is a (spherical) fusion category, so $\mathcal{Z}(\mathcal{TY}(A, \chi, \tau))$ is a modular category.

Properties of $\mathcal{Z}(\mathcal{TY}(A,\chi,\tau))$

Remarks

$\mathcal{Z}(\mathcal{TY}(A,\chi,\tau))$

- has simple objects of dimensions $1, 2$ and $\sqrt{|A|}$,
- is weakly integral,
- is not always group-theoretical when integral (i.e. when $\sqrt{|A|} \in \mathbb{Z}$),
- has rank $\frac{|A|(|A|+7)}{2}$,
- $\mathbb{Z}_2$-graded: $\mathcal{Z}(\mathcal{TY}(A,\chi,\tau)) = \mathcal{ZTY}(A,\chi,\tau)_+ \oplus \mathcal{ZTY}(A,\chi,\tau)_-$

First Results

Property **F** for $\mathcal{Z}(\mathcal{TY}(A,\chi,\tau))$ is mostly open.

Results

- $\mathcal{ZTY}(A,\chi,\tau)_+$ is group-theoretical (so has prop. **F**) [Naidu,R]
- $\mathcal{Z}(\mathcal{TY}(A,\chi,\tau))$ is group-theoretical iff $L = L^\perp$ for some subgroup $L \subset A$.

Weakly Group Theoretical Categories

Definition

- $\mathcal{D}$ is nilpotent if $\mathcal{D}_{ad} \supset (\mathcal{D}_{ad})_{ad} \supset \cdots$ converges to Vec.
- $\mathcal{C}$ is weakly group theoretical if $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}\mathcal{D}$ for $\mathcal{D}$ *nilpotent*.

- $\mathcal{C}$ weakly group theoretical $\Rightarrow$ $\mathcal{C}$ weakly integral
- Conjecturally, $\Leftarrow$, so
- Do weakly group theoretical categories have property **F**?

Related Problems

Question

- If $\mathcal{C}$ has property **F**, does $\mathcal{Z}(\mathcal{C})$ also?
- Do braided nilpotent categories have property **F**? (known if $\mathcal{C}$ is integral)
- Description of braiding?

Braided Vector Spaces

Let $R \in M_{m^2}(\mathbb{C})$ be a unitary solution to:
$R_1 R_2 R_1 = R_2 R_1 R_2$ where $R_1 = (R \otimes I)$ and $R_2 = (I \otimes R)$ and R has finite order.

Question

Image of $\mathcal{B}_n \to U(\mathbb{C}^{mn})$ finite?

Results

- If R comes from $D^\omega G$: Yes.
- For $m = 2$: Yes [Franko,R,Wang '05], [Franko, Thesis].

Conversely...

$\Psi_X : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$ "non-local" while for $X \in \mathrm{Rep}(D^\omega G)$ $\mathcal{B}_n$ acts locally on $X^{\otimes n}$.

Definition

Say Ψ_X can be unitarily localized if there is a unitary R-matrix R and a v.s. V so that $\Psi_X(\mathcal{B}_n)$ is realized as $\mathcal{B}_n$ acting on $V^{\otimes n}$ via R.

Fact

Reps. from $\mathcal{C}(\mathfrak{sl}_2, 4)$ [Franko,R,Wang '05] and $\mathcal{C}(\mathfrak{sl}_2, 6)$ can be unitarily localized.
and are weakly integral with property **F**.

Question (Wang)

Unitarily localized iff property **F**?

Link Invariants

If $\mathcal{C}$ is a ribbon fusion category, $X \in \mathcal{C}$, L a link:

$$\mathcal{I}_X(L) := tr_{\mathcal{C}}(\Psi_X(\beta))$$

is a $\mathbb{C}$-valued link invariant, where $\hat{\beta} = L$.

Question

Is computing (i.e. approximating, probabilistically) $\mathcal{I}_X(L)$ easy (polynomial-time) or hard (NP, assuming $P \neq NP$!)?

Appears to coincide with: *Is* $\Psi_X(\mathcal{B}_n)$ *finite or infinite?*
Related to topological quantum computers: weak or powerful?
(original motivation of Freedman, *et al*).

Thank You!