Localizing Unitary Braid Representations

Eric Rowell

Texas A&M University

Fudan University July 2010

Outline

Quantum Circuit Model

Let $V = \mathbb{C}^d$ d$\in \mathbb{Z}$ and $\mathcal{H}_{qc}(n) = V^{\otimes n}$($n$ qu-dit register)
Suppose $S := \{U_i\} \subset \mathbf{U}(\mathcal{H}_{qc}(n_i))$.
S is **a gate set**. Fix $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$.

Definition

A **quantum circuit** for U is:

- $G_1, \ldots, G_m \in \mathbf{U}(\mathcal{H}_{qc}(n))$
- where $G_i = I_V^{\otimes a} \otimes U_j \otimes I_V^{\otimes b}$ and
- $G_1 \cdot G_2 \cdots G_m = U$

Remarks on QC Model

Remarks

Realistically:

- S is finite
- $U_i \in \mathbf{U}(\mathcal{H}_{qc}(n_i))$ and $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$ with $n_i \ll n$
- S *approximately simulates*: $||G_1 \cdots G_m - U|| < \epsilon$
- S universal if any U is approx. simulated (with any $\epsilon > 0$).

Topological Phases of Matter

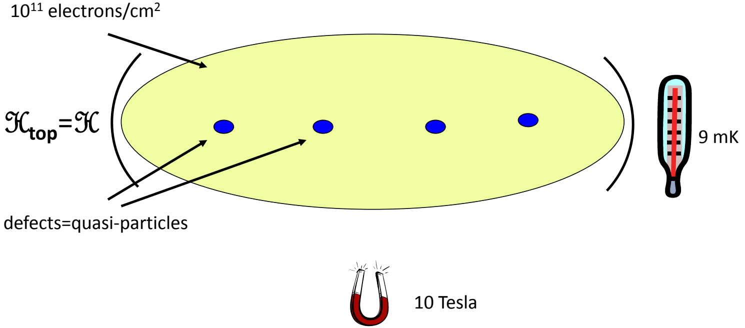
Definition (Das Sarma, et al)

a system is in a topological phase if its low-energy effective field theory is a *topological quantum field theory.*

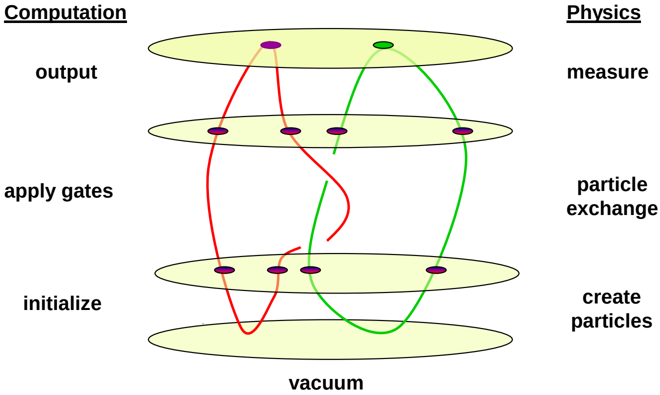
Topological quantum computers are realized via topological states of matter.

Example: FQH Liquid Cartoon

Fractional Quantum Hall Liquid



Topological Model



The Braid Group

Definition

$\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$

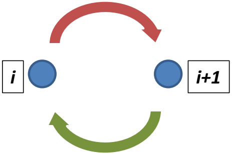
Remarks on Topological Model

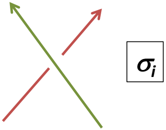
Remarks

- $\mathcal{H}_{top}(n)$ is *n-particle state space*
- Gate set S is "particle exchanges"
- Mathematically, $S = \{\varphi_n(\sigma_i)\}$ where

$$\varphi_n : \mathcal{B}_n \to \mathbf{U}(\mathcal{H}_{top}(n))$$

- $(\varphi_n, \mathcal{H}_{top}(n))$ are unitary $\mathcal{B}_n$-representations.





Comparison of Models

- $\dim \mathcal{H}_{qc}(n) = d^n$
- Gates are local: G_i acts on $n_i \ll n$ qu-dits
- Problem: decoherence
- many algorithms
- finite universal gate sets known.

- $\dim \mathcal{H}_{top}(n) \neq c^n$
- Gates are global: $\varphi_n(\sigma_i)$ *smeared* across $\mathcal{H}_{top}(n)$
- Fault-tolerant
- few algorithms
- finite universal gate sets known.

Best of Both Worlds?

Question

Is there a model that is:

- Universal
- purely topological (fault-tolerant) and
- local?

Why? QC algorithms in fault-tolerant universal setting!

Yang-Baxter Equation

Fix a vector space V and
let $R \in \mathrm{End}(V \otimes V)$ be an invertible solution to:

$$(YBE) \qquad (R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R)$$

Then we have representations $\varphi_R : \mathcal{B}_n \rightarrow \mathbf{GL}(V^{\otimes n})$ via
$\varphi_R(\sigma_i) = I^{\otimes i-1} \otimes R \otimes I^{\otimes n-i-1}$
Unitary if $R \in \mathbf{U}(V \otimes V)$.
Observe: representations are local: $\varphi_n(\sigma_i)$ acts on tensor factors $i, i+1$.

Unitary R-matrices

Remarks

- Solutions in literature usually not unitary
- Quantum groups: many more (non-unitary!) solutions.
- 4×4 unitary R-matrices classified (Dye,Hietaranta).
- unitary solutions from Hopf algebras $D(G)$ (G finite group).

Example

Example

Let $V = \mathbb{C}^2$ and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $\varphi_R : \mathcal{B}_n \to \mathbf{U}(V^{\otimes n})$.

Notice: $|\varphi_R(\mathcal{B}_n)| < \infty$: finite group.

Local, Unitary and Universal?

Difficulty

No **known** (finite order) unitary R-matrices such that $\varphi_R(\mathcal{B}_n)$ infinite.

So universality hard... Try something else.

$\mathcal{B}_n$-reps. from quotients

Let K be a field and $K\mathcal{B}_n$ the group algebra.
Let $I_n \subset K\mathcal{B}_n$ be a sequence of ideals such that

- $\dim(K\mathcal{B}_n/I_n) < \infty$
- $K\mathcal{B}_n/I_n \subset K\mathcal{B}_{n+1}/I_{n+1}$
- $K\mathcal{B}_n/I_n \cong \bigoplus_i \mathrm{End}(V_i^{(n)})$ (semisimple)
- and $\beta + I_n$ invertible for $\beta \in \mathcal{B}_n$.

Pull-back Reps.

Then $\pi_n : \mathcal{B}_n \to \prod_i \mathbf{GL}(V_i^{(n)})$.

Temperley-Lieb Algebras

Definition

The **Temperley-Lieb algebra** $TL_n(q)$ generators $e_1, \ldots, e_{n-1}$ satisfy:

(T1) $e_i^2 = e_i$

(T2) $e_i e_{i+1} e_i = \frac{1}{\beta} e_i$ where $\beta = 2 + q + 1/q$

(T3) $e_i e_j = e_j e_i$ if $|i - j| > 1$

(M) $\mathrm{tr}(e_{n-1} w) = \frac{1}{\beta} \mathrm{tr}(w)$ if $w \in TL_{n-1}(q)$.

Key fact: $\mathbb{C}(q)\mathcal{B}_n \twoheadrightarrow TL_n(q)$ by $\sigma_i \to g_i := (q+1)e_i - 1$.

Unitary Quotients

Let $q = e^{2\pi i/\ell}$.

Definition

$\overline{TL}_n(q) := TL_n(q)/Ann(\text{tr})$ is semisimple:

$$\overline{TL}_n(q) \cong \bigoplus_i \text{End}(V_i^{(n)})$$

$W^{(n)} := \bigoplus_i V_i^{(n)}$ faithful $\overline{TL}_n(q)$-module.

Defining $e_i^* = e_i$ we see $\overline{TL}_n(q)$ is a C^*-algebra.

Jones Representations

$$\mathbb{C}(q)\mathcal{B}_n \twoheadrightarrow \overline{TL}_n(q) \cong \bigoplus_i \mathrm{End}(V_i^{(n)})$$

induces unitary **Jones representations**:

Definition

$\rho_n^{(\ell)} : \mathcal{B}_n \rightarrow \prod_i \mathbf{U}(V_i^{(n)})$

$V_i^{(n)}$ are irreducible, unitary $\mathcal{B}_n$-representations.

Relationship with FQHE

- Jones Representations related to $\mathbf{SU}(2)_k$ CFT
- Read-Rezayi Conjecture relates $\mathbf{SU}(2)_k$ to FQHE

Jones Rep. $\rho_n^{(\ell)}$	$\mathbf{SU}(2)_k$	FQHE ν
$\ell = 3$	$\mathbf{SU}(2)_1$	$\nu = 1/3$ (abelian)
$\ell = 4$	$\mathbf{SU}(2)_2$	$\nu = 5/2$ (not universal)
$\ell = 5$	$\mathbf{SU}(2)_3$	$\nu = 13/5$ (universal)

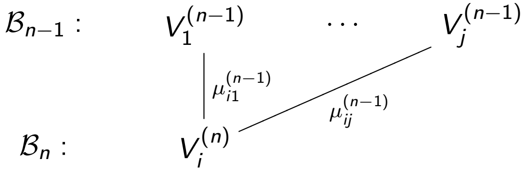
Bratteli Diagrams

Let $V_i^{(n)}$ $\mathcal{B}_n$-subrepresentation of $\rho_n^{(\ell)}$.Restrict $V_i^{(n)}$ to $\mathcal{B}_{n-1}$.
How does $V_i^{(n)}$ decompose as $\mathcal{B}_{n-1}$-rep?

$$V_i^{(n)} \cong \bigoplus_j \mu_{ij}^{(n-1)} V_j^{(n-1)}$$

Recorded in **Inclusion Matrix** $G^{(n-1)} := [\mu_{ij}^{(n-1)}]_{ij}$ or

Bratteli diagram



Example: $\overline{TL}_n(i)$

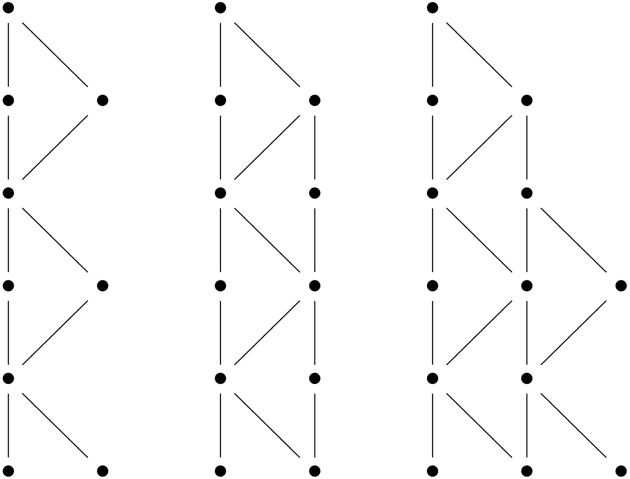
Example

$$\overline{TL}_n(i) \cong \begin{cases} \mathrm{End}(V_1^{(n)}) \oplus \mathrm{End}(V_2^{(n)}) & n \quad \text{even} \\ \mathrm{End}(V_1^{(n)}) & n \quad \text{odd} \end{cases}$$

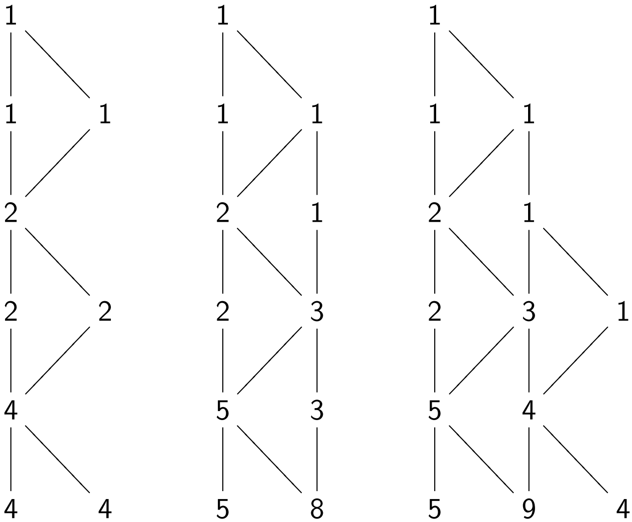
where $\dim V_1^{(n)} = \dim V_1^{(n+1)} = \dim V_1^{(n+1)} = 2^{n-2}$We have restriction/induction rules:

$$V_1^{(2k+1)} |_{\mathcal{B}_{2k}} = V_1^{(2k)} \oplus V_1^{(2k)}, \quad V_1^{(2k)} |_{\mathcal{B}_{2k-1}} = V_1^{(2k-1)}$$

Bratteli Diagrams $\ell = 4, 5, 6$



$\mathcal{B}_n$-rep. Dimensions $\ell = 4, 5, 6$



Characterizations

Let $q = e^{2\pi i/\ell}$ and $n > 10$.

Theorem (Jones 1986)

$\rho_n^{(\ell)}(\mathcal{B}_n)$ is infinite if, and only if $\ell \notin \{3, 4, 6\}$.

Theorem (Freedman,Larsen,Wang 2001)

$\rho_n^{(\ell)}(\mathcal{B}_n)$ is dense in $\prod_i \mathbf{SU}(V_i^{(n)})$ if, and only if $\ell \notin \{3, 4, 6\}$.

A Localization

Let $R = \alpha \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ with $\alpha = -\frac{e^{-\pi i/4}}{\sqrt{2}}$.

Theorem (Franko,R,Wang)

$g_i \to \varphi_R(\sigma_i)$ is a faithful $\overline{TL}_n(q)$-rep. for all n.

For Jones reps. with $\ell = 4$:

Localization

There exists a surj. $S : (\mathbb{C}^2)^n \twoheadrightarrow W^{(n)} = \bigoplus_i V_i^{(n)}$ such that $S\varphi_R(\beta) = \rho_n^{(4)}(\beta)S$ for all $\beta \in \mathcal{B}_n$.

Local Jones Reps.?

Question

Can we "localize" *all* Jones reps. $\rho_n^{(\ell)}$?
That is, find a unitary solution to (YBE) (R, V) such that:

- $(\varphi_R, V^{\otimes n})$ decomposes as

$$V^{\otimes n} \cong \bigoplus_i m_{n,i} V_i^{(n)}$$

- with $m_{n,i} > 0$ for each i(so $W^{(n)} \subset V^{\otimes n}$).

For $\ell = 3, 4$: Yes! Call such a pair (R, V) a **localization** of $\rho_n^{(\ell)}$.

Combinatorial Consequences

Consider $(\rho_n^{(\ell)}, W^{(n)})$ denote $d_i^{(n)} := \dim V_i^{(n)}$ and $d^{(n)} := (d_i^{(n)})_i$ *dimension vector*
If (R, V) is a localization of $(\rho_n^{(\ell)}, W^{(n)})$ and

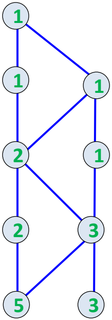
$$V^{\otimes n} \cong \bigoplus_i m_{n,i} V_i^{(n)}$$

denote $m^{(n)} = (m_{n,i})_i$ *localization vector* and $k = \dim(V)$.
We have:

Combinatorial Localization

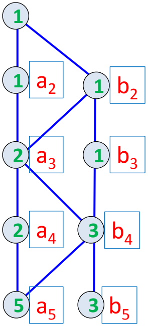
1. $k^n = \langle d^{(n)}, m^{(n)} \rangle$ (since $\dim(V^{\otimes n}) = k^n$)
2. $G^{(n-1)} m^{(n)} = k \cdot m^{(n-1)}$ (respects restriction).

Illustration: $\ell = 5$



If (R, V) localizes $\rho_n^{(5)}$ with $\dim(V) = k$

Illustration: $\ell = 5$



If (R, V) localizes $\rho_n^{(5)}$
with $\dim(V) = k$
and mult. vectors (a_n, b_n)
we must have:
$k^2 = a_2 + b_2$,
$k^3 = 2a_3 + b_3$, etc.
$ka_2 = a_3$,
$kb_2 = a_3 + b_3$
$ka_3 = a_4 + b_4$,
$kb_3 = b_4$, etc.

Result

Theorem (R,Wang)

Unitary Jones reps. $(\rho_n^{(\ell)}, \bigoplus_i V_i^{(n)})$ *localizable*
if, and only if $\ell \notin \{3,4,6\}$

Proof.

(Sketch) Graph theoretical techniques:
Perron-Frobenius Theorem $\Rightarrow$ no combinatorial localization. □

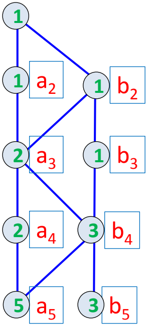
Perron-Frobenius Theorem

Theorem

If A is symmetric and non-negative
and λ is the maximal eigenvalue then there exists an eigenvector
$v > 0$
such that

$$\lim_{n\to\infty}(\frac{1}{\lambda}A)^n = vv^T$$

Example: $\ell = 5$

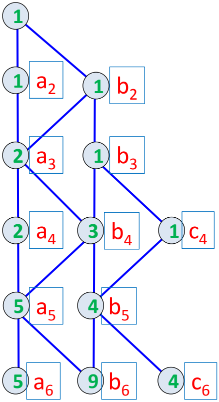


If (R, V) localizes $\rho_n^{(5)}$ with $\dim(V) = k$ and mult. vectors (a_n, b_n) Perron-Frobenius Theorem $\Rightarrow$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \tau \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Impossible! $(a_2, b_2 \in \mathbb{Z})$.

Example: $\ell = 6$



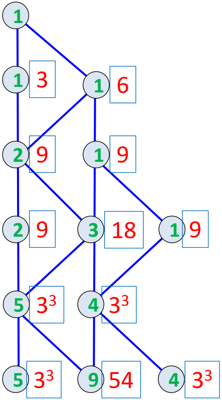
If (R, V) localizes $\rho_n^{(6)}$ with $\dim(V) = k$, mult. vectors (a_n, b_n, c_n)
Perron-Frobenius Theorem $\Rightarrow$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \lambda \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

$k = \lambda = 3$, $\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} z \\ z \end{pmatrix}$ works!

(Combinatorial Localization)

Example: $\ell = 6$



Is there a 9×9 *R-matrix?*

$$\gamma \begin{pmatrix} \omega & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega \\ 0 & \omega & 0 & 0 & 0 & \omega & 1 & 0 & 0 \\ 0 & 0 & \omega & \omega^2 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 & \omega & 0 & 0 & 0 & \omega^2 & 0 \\ \omega & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \omega & \omega & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 1 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & \omega^2 & 0 & 0 & 0 & \omega & 0 \\ 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & \omega \end{pmatrix}$$

cf. [Sun,Wang,Hu,Zhao,Wang,Xue]

Localizes $\rho_n^{(6)}$.

YBE Conjecture

Conjecture (R,Wang)

Suppose $R \in \text{End}(V \otimes V)$ is:

- Unitary solution to (YBE)
- with $R^k = I$ some $k \in \mathbb{Z}$

Then $\varphi_R(\mathcal{B}_n)$ is finite for all n.

Remarks

- False if R non-unitary or infinite order.
- True if $\dim(V) = 2$
- True for R-matrices of Hopf algebras $D(G)$, G finite group [Etingof,R,Witherspoon].

Localization Conjecture

Conjecture (R,Wang)

Let $(\varphi_n, Y^{(n)})$ be any sequence of unitary braid reps. Then:

- φ_n is localizable if and only if
- $|\varphi_n(B_n)| < \infty$ if and only if
- eigenvalues λ_n of inclusion matrices $G^{(n)}$ satisfy $\lambda_n^L \in \mathbb{Z}$.

Remarks

- Last condition related to Frobenius-Perron dimension.
- Does combinatorial localization imply localizable? (R, V) exists?

Generalized YBE

[R,Zhang,Wu,Ge]: Generalized YBE:
Fix a vector space V ($\dim(V) = k$) and let $I_s := I_V^{\otimes s}$.
Let $R \in \mathrm{End}(V^{\otimes m})$ be an invertible solution to:

$$(GYBE) \quad (R \otimes I_s)(I_V^{\otimes s} \otimes R)(R \otimes I_s) = (I_s \otimes R)(R \otimes I_s)(I_s \otimes R)$$

in $\mathrm{End}(V^{\otimes m+s})$.
Then $R_i := I_s^{\otimes i-1} \otimes R \otimes I_s^{\otimes n-i-1}$ satisfies braid relation $(R1)$
but $(R2)$: $R_i R_j = R_j R_i$ must be checked!
to obtain braid representations.

Generalized Localization?

Localization conjecture may need to be modified to allow (GYBE) solutions...

Example

[R,Zhang,Wu,Ge]: $V = \mathbb{C}^2$:

$$R := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Satisfies (GYBE) with $m = 3$ and $s = 2$ in $\mathrm{End}(V^{\otimes 5})$ and $(R2)$.

yields $\mathcal{B}_n$ reps.

Braided Fusion Categories

Most general setting:
Let X be an object in a (unitary) braided fusion category $\mathcal{C}$.
braiding $c_{X,X} \in \mathrm{End}(X \otimes X)$ gives rise to

$$\varphi_X : \mathcal{B}_n \rightarrow \mathbf{GL}(\mathrm{End}(X^{\otimes n})).$$

Example

Sub-quotients of $Rep(U_q\mathfrak{g})$ with $q = e^{\pi i/\ell}$ give unitary reps.

Difficulty

$\mathcal{B}_n$-reps. from braided fusion categories not well-understood.

Expected Paradigm

Conjecture

Suppse X is an object in a unitary braided (ribbon) fusion category $\mathcal{C}$. Then the following are **equivalent**:

- $\text{FPdim}(X)^2 \in \mathbb{Z}$ (max. eig. of fusion matrix)
- $\varphi_X(\mathcal{B}_n)$ finite group
- φ_X is localizable
- Top. Quantum Comput. model not universal
- Link invariant $Inv_X(L)$ classical (low comp. complexity).

Thank You!