Modular Categories and Braid Group Representations

Eric Rowell

Texas A&M University

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Outline

Some Axioms

Definition

A fusion category $\mathcal{C}$ is a monoidal category that is:

- abelian: $\oplus$ and $\mathbb{C}$-linear: $\mathrm{Hom}(X, Y)$ a v.s.
- finite rank: simple classes $\{X_0 := \mathbf{1}, X_1, \ldots, X_{m-1}\}$
- semisimple: $X \cong \bigoplus_i \mu_i X_i$
- rigid: duals X^*, $b_X : \mathbf{1} \to X \otimes X^*$, $d_X : X^* \otimes X \to \mathbf{1}$
- compatibility...

Ribbon Structure

Definition

A braided fusion category has (a natural family of) isomorphisms:

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying, e.g.

$$c_{X,Y \otimes Z} = (\mathrm{Id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{Id}_Z)$$

Definition

In a *ribbon* category $*$ and $c_{X,Y}$ are compatible encoded in twists $\theta_X : X \to X$ inducing $V \cong V^{**}$.

Modular Categories

Ribbon categories have:

- Consistent graphical calculus: braiding, twists, duality maps represented by (braid-like) diagrams
- canonical trace: $\mathrm{tr}_{\mathcal{C}} : \mathrm{End}(X) \to \mathbb{C} = \mathrm{End}(\mathbf{1})$
- categorical dimension: $\mathrm{tr}_{\mathcal{C}}(Id_X) := \dim(X) \in \mathbb{R}^{\times}$

Definition

Let $S_{i,j} = \mathrm{tr}_{\mathcal{C}}(c_{X_j,X_i} c_{X_i,X_j})$, $0 \leq i,j \leq m-1$. $\mathcal{C}$ is modular if $\det(S) \neq 0$.

Sources of Modular Categories

Example

- semisimple subquotient $\mathcal{C}(\mathfrak{g},\ell)$ of $\mathrm{Rep}(U_q\mathfrak{g})$ at $q=e^{\pi i/\ell}$ is (often) modular. (Andersen-Reshetikhin-Turaev-Wenzl)
- twisted double $D^\omega G$ of finite group G: $\mathrm{Rep}(D^\omega G)$ is modular
- $\mathcal{C}$ a spherical fusion cat. then Drinfeld center $\mathcal{Z}(\mathcal{C})$ is modular (due to Müger).

Grothendieck Semiring

Definition

$Gr(\mathcal{C}) := (Obj(\mathcal{C})/\sim, \oplus, \otimes, \mathbf{1})$ a unital based ring.

- Define matrices $(N_i)_{k,j} := \dim \text{Hom}(X_i \otimes X_j, X_k)$
- Rep. $\varphi : Gr(\mathcal{C}) \rightarrow \text{End}(\mathbb{Z}^m)$

$$\varphi(X_i) = N_i$$

- Respects duals: $\varphi(X^*) = \varphi(X)^T$
- If $\mathcal{C}$ is braided, $Gr(\mathcal{C})$ is commutative

Frobenius-Perron Dimensions

Definition

- FPdim(X) is the largest eigenvalue of $\varphi(X)$
- $\mathrm{FPdim}(\mathcal{C}) := \sum_{i=0}^{m-1} \mathrm{FPdim}(X_i)^2$

(a) $\mathrm{FPdim}(X) > 0$

(b) $\mathrm{FPdim} : Gr(\mathcal{C}) \to \mathbb{C}$ is a unital homomorphism

(c) FPdim is unique with (a) and (b).

Some Properties

Definition

$\mathcal{C}$ is

- pointed if $\text{FPdim}(X) = 1$ for X simple
- integral if $\text{FPdim}(X) \in \mathbb{Z}$ for all X (iff $\mathcal{C} \cong \text{Rep}(H)$, H s.s., f.d. quasi-Hopf algebra)
- weakly integral if $\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$ (iff $\text{FPdim}(X)^2 \in \mathbb{Z}$ for X simple)

Example

$\text{Rep}(DG)$ is integral, and pointed iff G abelian. Call $\mathcal{C}$ group theoretical if $\mathcal{Z}(\mathcal{C}) \cong \text{Rep}(D^\omega G)$ for some G.

Applications

Modular categories are of broad interest:

1. in Math. Phys.: related to Rational Conformal Field Theory
2. in Topology: 3D Topological Quantum Field Theories, link invariants.
3. in Condensed Matter Phys.: algebraic model for Topological Phases of Matter
4. in Algebra: Representation Theory of Hopf algebras, quantum groups, Iwahori-Hecke algebras...
5. in Quantum Computing: yes, really!

Classification Problems

Problems

1. Classify modular categories of fixed (low) rank. Wang's conjecture: there are finitely many for each rank.
2. Taxonomy: Stratify modular categories by properties: e.g. pointed $\subset$ group-theoretical $\subset$ integral $\subset$ weakly integral
3. Compare properties. *This Talk's Subject*

The Braid Group

Definition

$\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$

$\mathcal{B}_n$ reps. from BFCs

Example

Let $\mathcal{C}$ be a braided fusion category.

- Let $X \in \mathcal{C}$
- braiding $c_{X,X} \in \mathrm{End}(X \otimes X)$ gives rise to
  $\rho^X : \mathbb{C}\mathcal{B}_n \rightarrow \mathrm{End}(X^{\otimes n})$ by:
  $\rho^X(\sigma_i) = I_X^{\otimes i-1} \otimes c_{X,X} \otimes I_X^{\otimes n-i-1} \in \mathrm{End}(X^{\otimes n})$
- ρ^X induces $\mathcal{B}_n$-reps $(\rho^X, \mathrm{End}(X^{\otimes n}))$.

Note

$\mathcal{B}_n$ acts linearly on $\mathrm{End}(X^{\otimes n})$, not on $X^{\otimes n}$.

Example: Lie Type A_1

Example

$\mathcal{C}(\mathfrak{sl}_2, \ell)$ has rank $\ell - 1$ and:

- $\dim(X_k) = \text{FPdim}(X_k) = \frac{\sin(\frac{(k+1)\pi}{\ell})}{\sin(\frac{\pi}{\ell})}$
- $X_1 \otimes X_k = X_{k-1} \oplus X_{k+1}$
- is weakly integral iff $\ell \in \{2, 3, 4, 6\}$.
- ρ^{X_1} is the Jones representation $(\rho_n^{(\ell)}, \bigoplus_i V_n^{(i)})$.
- Associated link invariant: Jones polynomial

Broader Context

Let $\iota : \mathcal{B}_n \to B_{n+1}$, $\iota(\sigma_i) = \sigma_i$ for $i \leq n-1$.

Definition

A sequence of braid representations is a family of $\mathcal{B}_n$-reps (ρ_n, V_n) and maps τ_n such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & \xrightarrow{\rho_n} & \mathbb{C}\rho_n(\mathcal{B}_n) \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle \tau_n} \\ \mathbb{C}\mathcal{B}_{n+1} & \xrightarrow{\rho_{n+1}} & \mathbb{C}\rho_{n+1}(\mathcal{B}_{n+1}) \end{array}$$

Assumption

To simplify, assume $\mathcal{B}_n$-reps. (ρ_n, V_n) are unitary.

First Question

Question

Given $X \in \mathcal{C}$ is $\rho^X(\mathcal{B}_n)$ finite or infinite?

Definition

Say $\mathcal{C}$ has property **F** if $|\rho^X(\mathcal{B}_n)| < \infty$ for all X and n.

Remark

- Typically, if $\mathcal{C}$ does not have property **F**, then $|\rho^X(\mathcal{B}_3)| = \infty$.
- If X generates $\mathcal{C}$ (any simple Y has $Y \subset X^{\otimes m}$ for some m) then $|\rho^X(\mathcal{B}_n)| < \infty$ for all $n \Rightarrow \mathcal{C}$ has property **F**.

Examples

Theorem (Jones '86)

$\mathcal{C}(\mathfrak{sl}_2, \ell)$ *has property* **F** *iff* $\ell \in \{2, 3, 4, 6\}$.

Theorem (Etingof,R.,Witherspoon '08)

$\mathcal{C}$ *group-theoretical has property* **F** *(e.g.* $\mathcal{C} = \text{Rep}(D^\omega G)$*).*

Theorem (Rowell '10)

non-trivial $\mathcal{C}(\mathfrak{g}_2, \ell)$ *has property* **F** *iff* $\ell = 8$.

Braided Vector Spaces

Definition

(R, V) is a **braided vector space** if $R \in Aut(V \otimes V)$ satisfies

$$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R)$$

Induces a sequence of $\mathcal{B}_n$-reps $(\rho^R, V^{\otimes n})$ by

$$\rho^R(\sigma_i) = I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}$$

Example

Example

Let $V = \mathbb{C}^2$ and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $\rho^R : \mathcal{B}_n \to \mathbf{U}(V^{\otimes n})$.

Notice: $|\rho^R(\mathcal{B}_n)| < \infty$: finite group.

Second Question

Notice that $(\rho^R, V^{\otimes n})$ is local: $\rho^R(\sigma_i)$ acts non-trivially only on adjacent tensor factors:

$$v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \xrightarrow{\rho^R(\sigma_i)} v_1 \otimes \cdots \otimes R(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n$$

Question

Given a sequence (ρ_n, V_n), when can it be realized via braided v.s. (R, V)? "Localized"

Formal Definition

Definition

A localization of a sequence of unitary $\mathcal{B}_n$-reps. (ρ_n, V_n) is a braided vector space (R, W) such that for all $n \geq 2$:

(i) there exist $\varphi_n : \mathbb{C}\rho_n(\mathcal{B}_n) \to \mathrm{End}(W^{\otimes n})$ such that the following diagram commutes for all n:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & & \\ \downarrow{\scriptstyle \rho_n} & \searrow^{\rho^R} & \\ \mathbb{C}\rho_n(\mathcal{B}_n) & \xrightarrow{\varphi_n} & \mathrm{End}(W^{\otimes n}) \end{array}$$

(ii) and $(\varphi_n, W^{\otimes n})$ is a faithful $\mathbb{C}\rho_n(\mathcal{B}_n)$-module.

In other words...

If (R, W) localizes (ρ_n, V_n),

- Decompose (ρ_n, V_n): $V_n \cong \bigoplus_{i \in J_n} V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- then $W^{\otimes n} \cong \bigoplus_{i \in J_n} \mu_n^i V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- with $\mu_n^i > 0$ (multiplicities)

Remarks

- $\dim(V_n) \neq d^n$ (usually), so extra copies of some $V_n^{(i)}$ needed.
- (R, W) uniformly localizes for all n.
- $\overrightarrow{\mu}_n$ localization vector.

Motivating Example: $\mathcal{C}(\mathfrak{sl}_2, 4)$

Let $R = \alpha \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ with $\alpha = -\frac{e^{-\pi i/4}}{\sqrt{2}}$.

Theorem (Franko,R,Wang '06)

$(R, \mathbb{C}^2)$ localizes Jones representation $\rho_n^{(4)}$ associated with $X_1 \in \mathcal{C}(\mathfrak{sl}_2, 4)$.

Recall: $\mathcal{C}(\mathfrak{sl}_2, 4)$ is weakly integral

First Results

Theorem (R,Wang)

Jones reps. $(\rho_n^{(\ell)}, \bigoplus_i V_i^{(n)})$ *localizable if, and only if* $\ell \in \{2, 3, 4, 6\}$

Theorem (Easy)

$(\rho^H, \mathrm{End}(H^{\otimes n}))$ *localizable for* $H = DG$.

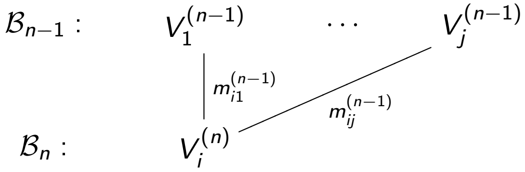
Bratteli Diagrams

Consider irreducible $\mathcal{B}_n$-rep $V_i^{(n)}$.
How does $V_i^{(n)}|_{\mathcal{B}_{n-1}}$ decompose?

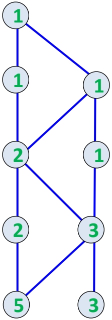
$$V_i^{(n)} \cong \bigoplus_j m_{ij}^{(n-1)} V_j^{(n-1)}$$

Recorded in **Inclusion Matrix** $G^{(n-1)} := [m_{ij}^{(n-1)}]_{ij}$ or

Bratteli diagram

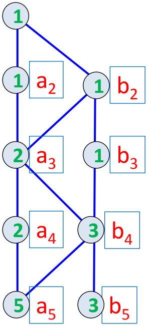


Example: $\mathcal{C}(\mathfrak{sl}_2, 5)$



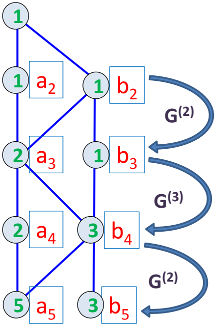
If (R, V) localizes $\rho_n^{(5)}$

Example: $\mathcal{C}(\mathfrak{sl}_2, 5)$



If (R, V) localizes $\rho_n^{(5)}$
with mult. vectors (a_n, b_n)

Example: $\mathcal{C}(\mathfrak{sl}_2, 5)$



If (R, V) localizes $\rho_n^{(5)}$ with mult. vectors (a_n, b_n) then by Perron-Frobenius Theorem

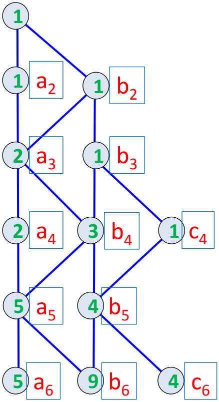
$$G^{(3)} G^{(2)} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \lambda \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

where $G^{(3)} G^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$\lambda = \mathrm{FPdim}(X_1)^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2$, $a_2, b_2 \in \mathbb{Z}$.

Impossible!

Example: $\mathcal{C}(\mathfrak{sl}_2, 6)$



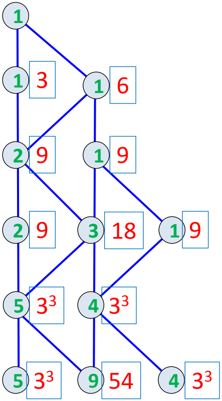
If (R, V) localizes $\rho_n^{(6)}$ with $\dim(V) = k$ then

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_4 \\ b_4 \\ c_4 \end{pmatrix} = \lambda \begin{pmatrix} a_4 \\ b_4 \\ c_4 \end{pmatrix}$$

and $2a_4 + 3b_4 + c_4 = k^4$

$k = \lambda = 3,\ a_4 = b_4/2 = c_4 = 9$ works!

Example: $\mathcal{C}(\mathfrak{sl}_2, 6)$



Is there a 9×9 *R-matrix?*

$$\gamma \begin{pmatrix} \omega & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega \\ 0 & \omega & 0 & 0 & 0 & \omega & 1 & 0 & 0 \\ 0 & 0 & \omega & \omega^2 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 & \omega & 0 & 0 & 0 & \omega^2 & 0 \\ \omega & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \omega & \omega & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 1 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & \omega^2 & 0 & 0 & 0 & \omega & 0 \\ 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & \omega \end{pmatrix}$$

Localizes $\rho_n^{(6)}$.

Braided Vector Space Conjecture

Conjecture (R,Wang)

Suppose (R, V) is a braided v.s. with:

- R Unitary
- R finite order $(R^k = I)$

Then $\rho^R(\mathcal{B}_n)$ is finite for all n.

Remarks

- True if $\dim(V) = 2$.
- True for (R_V, V) where $V \in \mathrm{Rep}(D^\omega G)$.
- Replace unitary by algebraic condition?

Localization Conjecture

Conjecture (R,Wang)

For unitary $(\rho^X, \mathrm{End}(X^{\otimes n}))$ TFAE:

(L) ρ^X is localizable, with R finite order

(F) $|\rho^X(B_n)| < \infty$

(W) $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$

(P) Link invariants efficiently computable on Turing Machine.

Remarks

- (F) $\Leftrightarrow$ (W) is the *property* **F** *conjecture* (R. '07)
- (F) $\Leftrightarrow$ (P) suggested in (Larsen,R,Wang '05).

Some Evidence

Theorem (R,Wang)

Suppose that $\mathbb{C}\rho^X(\mathcal{B}_n) = \mathrm{End}(X^{\otimes n})$ *for all* n*, i.e.* $\rho^X(\mathcal{B}_n)$ *generates all of* $\mathrm{End}(X^{\otimes n})$*. If* $(\rho^X, \mathrm{End}(X^{\otimes n}))$ *is localizable then* $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ *((L)* $\Rightarrow$ *(W)).*

Theorem

Localization conjecture true for $V \in \mathrm{Rep}(DG)$ *with* G *abelian or dihedral, and for* $X_1 \in \mathcal{C}(\mathfrak{sl}_2, \ell)$*.*

Current Work

localizable may be too strong. Two weaker conditions:

1. Generalized localization: $R \in Aut(V^{\otimes k})$ such that: $\sigma_i \longrightarrow I_V^{\otimes m(i-1)} \otimes R \otimes I_V^{m(n-i-1)}$ defines $\mathcal{B}_n$-rep.
2. quasi-localization: via quasi-braided vector spaces (R, Φ, V).

Current joint work with S.-M. Hong and C. Galindo.

Thank You!