Topological Quantum Computation

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Outline

The State Space

Fix d$\in \mathbb{Z}$

Definition

Let $V = \mathbb{C}^d$. The n-qudit **state space** is the n-fold tensor product:

$$\mathcal{H}_{qc}(n) = V \otimes V \otimes \cdots \otimes V.$$

$|v_1\rangle \otimes |v_2\rangle \cdots \otimes |v_n\rangle \in \mathcal{H}_{qc}(n)$ is a **n qudit register**.

Notice: $\dim \mathcal{H}_{qc}(n) = d^n$.

Gates and Circuits

A **quantum gate** is a unitary operator $U_i \in \mathbf{U}(\mathcal{H}_{qc}(n_i))$
A **gate set** $S = \{U_i\}$ is a collection of gates.
Fix $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$.

Definition

A **quantum circuit** for U on S is:

- $G_1, \ldots, G_m \in \mathbf{U}(\mathcal{H}_{qc}(n))$
- where $G_i = I_V^{\otimes a} \otimes U_j \otimes I_V^{\otimes b}$, $U_j \in S$, $a + b + n_i = n$ and
- $G_1 \cdot G_2 \cdots G_m = U$

Remarks

Remarks

Realistically:

- $U_i \in \mathbf{U}(\mathcal{H}_{qc}(n_i))$ and $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$ with $n_i \ll n$ so G_i are **local**.
- S is finite so for fixed n only finitely many G_i possible.

Thus

Sad Fact

No finite gate set can exactly implement every $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$.

Universality

Let $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$ and S a gate set.

Definition

S **approximately simulates** U within ϵ if there exists $G_1, \cdots, G_m$ such that $||G_1 \cdots G_m - U|| < \epsilon$

Here $|| \cdot ||$ is the operator norm.

Definition

If S approximately simulates any $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$ within ϵ for any $\epsilon > 0$ S is **universal**.

In other words, if the set of promotions of S to $\mathbf{U}(\mathcal{H}_{qc}(n))$ is **dense**.

Universal Examples

The following set is universal for d$= 2$: $\{H, \sigma_z^{\pm 1/4}, CNOT\}$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\sigma_z^{\pm 1/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pm \pi i/4} \end{pmatrix}$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Also any **entangling** $U \in \mathbf{U}(\mathcal{H}_{qc}(2))$ with **all** of $\mathbf{U}(\mathcal{H}_{qc}(1))$ is univeral (Brylinskis' Theorem). (A tall order!)

Efficiency?

Let S be universal, $U \in \mathbf{U}(\mathcal{H}_{qc}(n))$ and $\epsilon > 0$ so that $||G_1 \cdots G_m - U|| < \epsilon$. How big is m?

Theorem (Solovay-Kitaev)

If S is closed under $U_i \rightarrow U_i^{-1}$ then $m = O(\log^2 1/\epsilon)$.

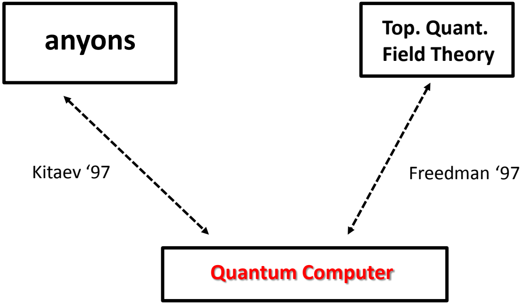
BQP

Class of problems efficiently solved by QC:

B(ounded error)Q(uantum resource)P(olynomial time)

Origins

Some History



Topological Phases of Matter

Definition (Nayak, et al)

a system is in a topological phase if its low-energy effective field theory is a *topological quantum field theory.*

Remarks

- Anyons are examples of topological phases
- Hence "Freedman-Kitaev" model: TQFTs model systems of anyons.
- a topological quantum computer would be realized on such systems.

Topological Quantum Field Theory?

Definition

A (unitary) 3D **TQFT** assigns to any (compact oriented labelled extended) surface (M, ℓ) a (finite-dimensional) Hilbert space: $\mathcal{H}_{top}(M, \ell)$, subject to (many) compatibility axioms. Key: gluing and disjoint sum axioms.

Labels $\mathcal{L}$ a finite set, $0 \in \mathcal{L}$ distinguished, with involution $x \to \hat{x}$.

Remarks

- Each component of boundary ∂M is labelled
- $\mathcal{H}_{top}(D^2, 0) = \mathbb{C}$.
- $\mathcal{H}_{top}(n) := \mathcal{H}_{top}(D^2 \setminus \{z_i\}_{i=1}^n, (0, t, \ldots, t))$

Two axioms

Axiom (Disjoint Sum)

$$\mathcal{H}_{top}((M_1,\ell_1)\coprod(M_2,\ell_2)) = \mathcal{H}_{top}(M_1,\ell_1)\otimes\mathcal{H}_{top}(M_2,\ell_2)$$

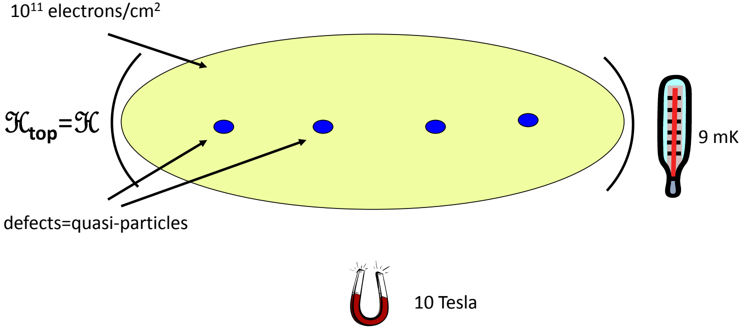
Axiom (Gluing)

If M_g is obtained from gluing two boundary circles of M together then

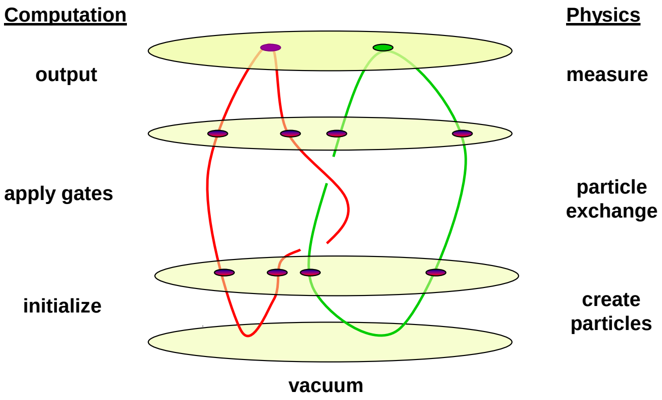
$$\mathcal{H}_{top}(M_g,\ell) = \bigoplus_{x\in\mathcal{L}}\mathcal{H}_{top}(M,(\ell,x,\hat{x}))$$

Example: FQH Liquid Cartoon

Fractional Quantum Hall Liquid



Topological Model (non-adaptive)



The Braid Group

A key role is played by: :

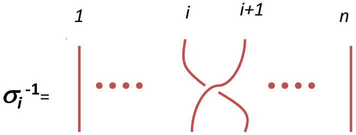
Definition

The **braid group** $\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

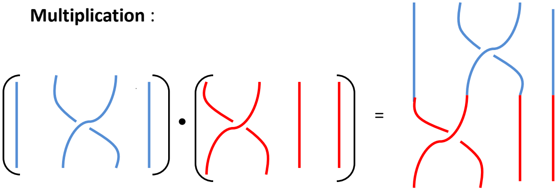
(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$

In Pictures: Source of Fault-Tolerence



Multiplication :



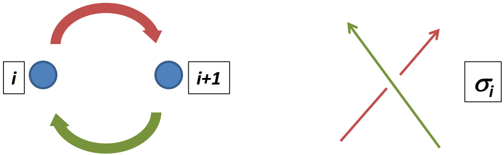
Remarks on Topological Model

Remarks

- $\mathcal{H}_{top}(n)$ is n-particle state space
- Gate set S is "particle exchanges"
- Mathematically, $S = \{\varphi_n(\sigma_i)\}$ where

$$\varphi_n : \mathcal{B}_n \to \mathbf{U}(\mathcal{H}_{top}(n))$$

- $(\varphi_n, \mathcal{H}_{top}(n))$ are unitary $\mathcal{B}_n$-representations.



Fibonacci Theory

Two particle types: $X_0 = \mathbf{1}$ (vacuum) and X_1.

- $X_1 \otimes X_1 = \mathbf{1} \oplus X_1$: two **fusion channels** (non-abelian!).
- $V_k^i := \mathcal{H}_{top}(D^2 \setminus \{z_i\}_{i=1}^k, (i, 1, \ldots, 1))$ ∂D^2 labeled by i
- $V_3^1 \cong \mathbb{C}^2$ (qubits!)
- More generally, $\dim V_k^i = \begin{cases} Fib(n-2) & i = 0 \\ Fib(n-1) & i = 1 \end{cases}$

  $1, 1, 2, 3, 5, 8, \ldots, \frac{\tau^n - (-\tau)^{-n}}{\sqrt{5}}, \ldots$

Why Universal?

Particle exchange induces the:

Definition

Jones representation (at 5th roots of unity):

$$\rho_n^5 : \mathcal{B}_n \to \mathbf{U}(V_n^0) \times \mathbf{U}(V_n^1) \subset \mathbf{U}(V_n^0 \oplus V_n^1)$$

Theorem (Freedman,Larsen,Wang)

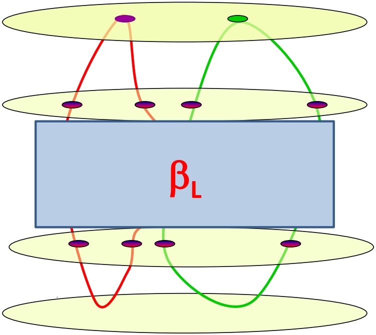
$\{\rho_n^5(\sigma_i)\}_{i=1}^{n-1}$ *is dense in* $\mathbf{SU}(V_n^0) \times \mathbf{SU}(V_n^1)$.

Universality follows from Kitaev-Solovay (after some comparisons...)

What do TQCs like to compute?

Answer

(Approximations to) Link invariants! (at roots of unity).
Fibonacci theory: Jones polynomial $J_L(q)$ at $q = e^{2\pi i/5}$.



Prob(●) ~ V_L(q)

Complexity of Jones Polynomial Evaluations

Classically we have:

Theorem (Vertigan,Wocjan-Yard)

Exact computation of $J_L(q)$ at $q = e^{2\pi i/r}$ is:

$$\begin{cases} FP & r = 3,4,6 \\ FP^{\#P} & else \end{cases}$$

And on a quantum computer:

Theorem

Approximation of $J_L(q)$ at $q = e^{2\pi i/r}$ is BQP-complete for $r \neq 3,4,6$.

Head-to-head

- State space a tensor product: $\dim \mathcal{H}_{qc}(n) = d^n$
- Gates are local: G_i acts on $n_i \ll n$ qu-dits in $\mathcal{H}_{qc}(n)$
- Problem: decoherence
- many algorithms

- State space not a tensor product: $\dim \mathcal{H}_{top}(n) \neq c^n$
- Gates are global: $\varphi_n(\sigma_i)$ *smeared* across $\mathcal{H}_{top}(n)$
- Fault-tolerant
- few algorithms

Equivalence

Theorem (Freedman,Kitaev,Wang)

(Universal) QCM simulates TQCs efficiently.

Theorem (Freedman,Larsen,Wang)

(at least one) TQC simulates QCM efficiently.

Main Issue

$\mathcal{H}_{qc}(n) \cong V^{\otimes n}$ while $\mathcal{H}_{top}(k) \cong \bigoplus_i W_n^i$

Mathematics of Equivalences

Question

- Given $U_f \in \mathbf{U}(\mathcal{H}_{qc}(n))$ how to (approximately) simulate U_f on $\mathcal{H}_{top}(k)$?
- Given $U_\beta \in \mathbf{U}(\mathcal{H}_{top}(k))$ how to simulate U_β on $\mathcal{H}_{qc}(n)$?

Answer

- **Efficiently** embed $V^{\otimes n}$ into $\mathcal{H}_{top}(k(n))$.
- **Efficiently** embed $\mathcal{H}_{top}(k)$ into $V^{\otimes n(k)}$.

Each uses gluing axiom

Inconvenient Truth

Leads to **leakage** errors. (next time...)

Best of Both Worlds?

Question

Is there a model that is:

- Universal
- purely topological (fault-tolerant) and
- (explicitly) local?

Why? QC algorithms in fault-tolerant universal setting!

Thank You!