Localizing Topological Quantum Computers

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Outline

1. QCM vs. TQC
    - Last Time...
    - Examples
    - Equivalences: Further Details
2. Localizing Braid Reps
    - Sequences of Braid Group Representations
    - Localization
    - Conjectures

Recall

Definition

A topological quantum computer is a computational model:

- based upon two-dimensional quantum systems of topological phases
- gates are achieved by *particle exchange*
- measurement is *fusion* of quasi-particles
- universality is (essentially) equivalent to *density* of braid group representations.
- Mathematically, modeled by a Topological Quantum Field Theory

Topological Quantum Field Theory?

Definition

A (unitary) 3D **TQFT** assigns to any (compact oriented labelled extended) surface (M,ℓ) a (finite-dimensional) Hilbert space: $\mathcal{H}(M,\ell)$, subject to (many) compatibility axioms. Key: gluing and disjoint sum axioms.

Labels $\mathcal{L}$ a finite set, $0 \in \mathcal{L}$ distinguished, with involution $x \rightarrow \hat{x}$.

Definition

The computational space is:$\mathcal{H}(n) := \mathcal{H}(D^2 \setminus \{z_i\}_{i=1}^{n}, (0, t, \dots, t))$

More Details

Each component of boundary ∂M is labelled by $i \in \mathcal{L}$

Remarks

Basic pieces are:

- empty: $\mathcal{H}(\emptyset) = \mathbb{C}$
- disk: $\mathcal{H}(D^2; i) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & else \end{cases}$
- annulus: $\mathcal{H}(A; a, b) = \begin{cases} \mathbb{C} & a = \hat{b} \\ 0 & else \end{cases}$
- pants: $P := D^2 \setminus \{z_1, z_2\}$ $\mathcal{H}(P; a, b, c) = \mathbb{C}^{N(a,b,c)}$

Two axioms

To determine $\mathcal{H}(M;\{i\})$ use basic pieces and two axioms:

Axiom (Disjoint Sum)

$$\mathcal{H}((M_1,\ell_1)\coprod(M_2,\ell_2))=\mathcal{H}(M_1,\ell_1)\otimes\mathcal{H}(M_2,\ell_2)$$

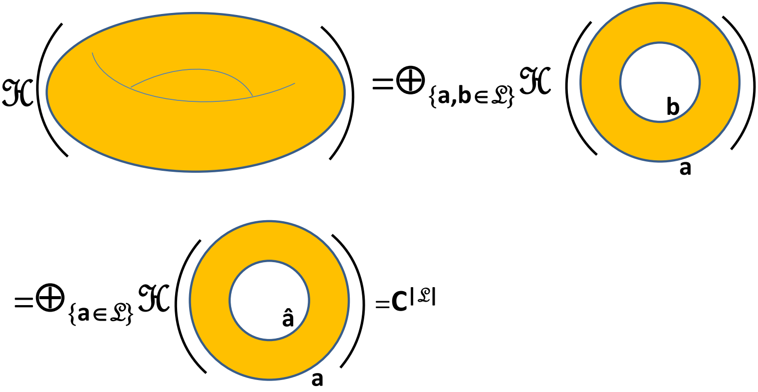
Axiom (Gluing)

If M_g is obtained from gluing two boundary circles of M together then

$$\mathcal{H}(M_g,\ell)=\bigoplus_{x\in\mathcal{L}}\mathcal{H}(M,(\ell,x,\hat{x}))$$

Example: Torus

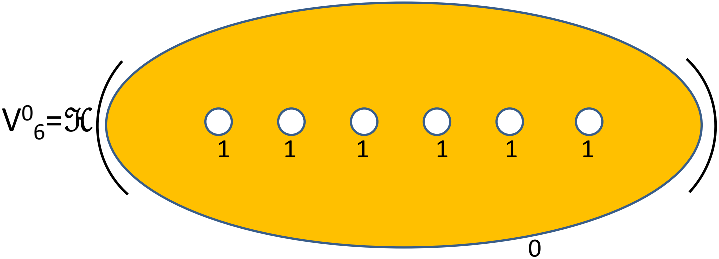
$$\mathcal{H}(T^2) = \mathbb{C}^{|\mathcal{L}|}:$$



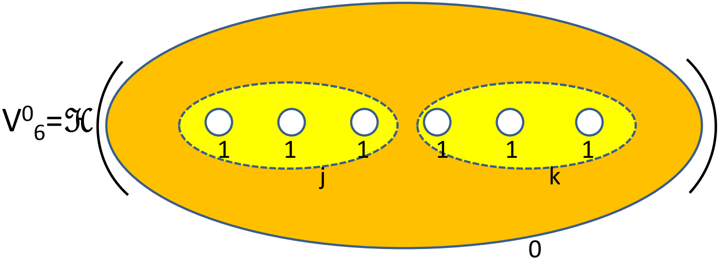
Example: Fibonacci Theory

- $\mathcal{L} = \{0, 1\}$
- pants: $\mathcal{H}(P; a, b, c) = \begin{cases} \mathbb{C} & a = b = c \\ \mathbb{C} & a + b + c \in 2\mathbb{Z} \\ 0 & else \end{cases}$
- Define: $V_k^i := \mathcal{H}(D^2 \setminus \{z_i\}_{i=1}^k; i, 1, \cdots, 1)$
- qubits realized on $V_3^1 \cong \mathbb{C}^2$.
- dim $V_n^i = \begin{cases} Fib(n-2) & i = 0 \\ Fib(n-1) & i = 1 \end{cases}$

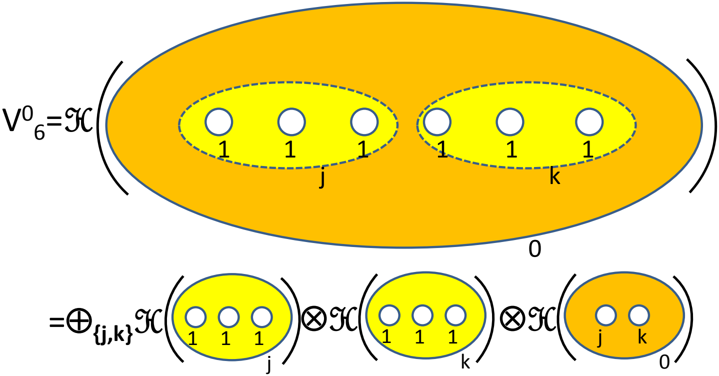
Example: V_6^0



Example: V_6^0



Example: V_6^0



$$= \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C} \oplus \mathbf{C} \otimes \mathbf{C} \otimes \mathbf{C} = \mathbf{C}^4 \oplus \mathbf{C}$$

Simulating QCM gates

Set $\mathcal{S}_k = (\mathbb{C}^2)^{\otimes k}$ QCM state space
Goal: simulate $U \in \mathbf{U}(\mathcal{S})$ on $\mathcal{H}(n(k))$

1. Set $V_3^1 \cong \mathbb{C}^2$
2. Identify $\mathcal{S}_k \cong (V_3^1)^{\otimes k}$
3. embed $\mathcal{S}_k \overset{\otimes v}{\hookrightarrow} \mathcal{S} \otimes V_k^0 = (V_3^1)^{\otimes k} \otimes V_k^0$ (pick $v \in V_k^0$, use disj. union axiom)
4. embed $(V_3^1)^{\otimes k} \otimes V_k^0 \hookrightarrow V_{3k}^0$ (as summand, via gluing)
5. So $i_v : \mathcal{S}_k \hookrightarrow V_{3k}^0$.
6. Now $i_v \circ U|x\rangle \approx \rho_{3k}(\beta_U) \circ i_v|x\rangle$

Issues

Remarks

- So to simulate 2-qubit gates, we need a 6-puntured disk
- From example: $\mathcal{S}_2 \hookrightarrow (\mathcal{S}_2) \oplus \mathbb{C} = V_6^0$
- $\mathbb{C}$ summand is non-computational
- leads to *leakage errors*: information leaks into $\mathbb{C}$ and is lost in measurement.
- This can be fixed but a "black art" generally...
- More efficient ways known: N. Bonesteel *et al.*

Simulating Fibonacci on QCM

Let $U(\beta) \in \mathbf{U}(V_n^1)$ be a unitary braiding matrix (for simplicity).
Goal: simulate U on $X^{\otimes k(n)}$ for some v.s. X.

1. Set $X = \bigoplus_{(a,b,c)\in\mathcal{L}} \mathcal{H}(P, a, b, c)$.
2. Decompose $(D^2 \setminus \{z_i\}_{i=1}^n; 1, \ldots, 1)$ into $n-1$ pairs of pants.
3. Set $W = \bigotimes^{(n-1)} X = \bigotimes^{(n-1)} \bigoplus_{(a,b,c)\in\mathcal{L}} \mathcal{H}(P; a, b, c)$
4. Distributing $\otimes$ over $\oplus$ gluing and disj. union axioms imply
$$V_n^1 \hookrightarrow W$$

Issues

Remarks

- $\dim(X) = 5$ so not very efficient
- $U(\beta)$ only acts on a subspace of $W = X^{\otimes n-1}$
- Must use so-called F-**matrices** encoding associativity: decompose β as product of $\sigma_i^{\pm 1}$, use different embeddings for odd/even i, conjugate them by F-matrices...

Best of Both Worlds?

Question

Is there a model that is:

- Universal
- purely topological (fault-tolerant) and
- (explicitly) local?

Why? QC algorithms in fault-tolerant universal setting!

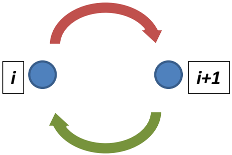
The Braid Group

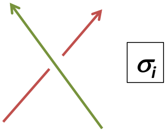
Definition

$\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$





Fibonacci Theory: Jones Reps.

Particle exchange induces the:

Definition

Jones representation (at 5th roots of unity):
$\rho_n^5 : \mathcal{B}_n \to \mathbf{U}(V_n^0) \times \mathbf{U}(V_n^1) \subset \mathbf{U}(V_n^0 \oplus V_n^1)$

Question

Can we find $\mathcal{B}_n$-representations $(\varphi_n, W^{\otimes n})$ *realizing* ρ_n^5?

Remarks

- $\dim(V_n^0 \oplus V_n^1) = Fib(n) \neq d^n$
- a *realization* must contain ρ_n^5 (faithful) and
- have only V_n^0 and V_n^1 as sub-reps.

Sequences of Braid Representations

Let $\iota : \mathcal{B}_n \to B_{n+1}$, $\iota(\sigma_i) = \sigma_i$ for $i \leq n-1$.

Definition

A **sequence of braid representations** is a family of $\mathcal{B}_n$-reps (ρ_n, V_n) and maps τ_n such that the following diagram commutes for all n:

$$\begin{tikzcd}
\mathbb{C}\mathcal{B}_n \arrow[r] \arrow[d, hook, "\iota"] & \mathbb{C}\rho_n(\mathcal{B}_n) \arrow[d, hook, "\tau_n"] \\
\mathbb{C}\mathcal{B}_{n+1} \arrow[r] & \mathbb{C}\rho_{n+1}(\mathcal{B}_{n+1})
\end{tikzcd}$$

Braided Vector Spaces

Definition

(R, V) is a **braided vector space** if $R \in Aut(V \otimes V)$ satisfies

$$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R)$$

Induces a sequence of $\mathcal{B}_n$-reps $(\rho_R, V^{\otimes n})$ where

$$\rho_R(\sigma_i) = I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}$$

Representations are local: $\rho_n(\sigma_i)$ acts on tensor factors $i, i+1$.

Example

Example

Let $V = \mathbb{C}^2$ and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $\rho_R : \mathcal{B}_n \to \mathbf{U}(V^{\otimes n})$.

Notice: $|\rho_R(\mathcal{B}_n)| < \infty$: finite group.

Formal Definition

Definition

Suppose (ρ_n, V_n) is a sequence of completely reducible braid representations. A **localization** of (ρ_n, V_n) is a braided vector space (W, R) such that for all $n \geq 2$:

(i) there exist $\varphi_n : \mathbb{C}\rho_n(\mathcal{B}_n) \to \mathrm{End}(W^{\otimes n})$ such that the following diagram commutes for all n:

$$\begin{tikzcd}
\mathbb{C}\mathcal{B}_n \arrow[d, "\rho_n"] \arrow[dr, "\rho_R"] & \\
\mathbb{C}\rho_n(\mathcal{B}_n) \arrow[r, "\varphi_n"] & \mathrm{End}(W^{\otimes n})
\end{tikzcd}$$

(ii) and $(\varphi_n, W^{\otimes n})$ is a faithful $\mathbb{C}\rho_n(\mathcal{B}_n)$-module.

In other words...

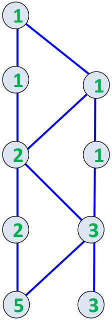
If (R, W) localizes (ρ_n, V_n),

- Decompose (ρ_n, V_n): $V_n \cong \bigoplus_{i \in J_n} V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- then $W^{\otimes n} \cong \bigoplus_{i \in J_n} \mu_n^i V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- with $\mu_n^i > 0$ (strictly positive multiplicities)

Remarks

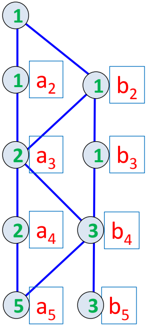
- $\dim(V_n) \neq d^n$ (usually), so extra copies of some $V_n^{(i)}$ needed.
- (R, W) uniformly localizes for all $n \geq 2$.
- $\overrightarrow{\mu}_n$ **localization vector**.

Illustration: Fibonacci



If (R, V) localizes $\rho_n^{(5)}$
with $\dim(V) = k$

Illustration: Fibonacci



If (R, V) localizes $\rho_n^{(5)}$ with $\dim(V) = k$ and mult. vectors (a_n, b_n) then

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \frac{1+\sqrt{5}}{2} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Impossible! $a_2, b_2 \in \mathbb{Z}$.

YBE Conjecture

Conjecture (R,Wang)

Suppose $R \in \mathrm{End}(V \otimes V)$ solution to (YBE) is:

- Unitary
- finite order ($R^k = I$)

Then $\rho_R(\mathcal{B}_n)$ is finite for all n.

Localization Conjecture

Conjecture (R,Wang)

Let (ρ_n, V_n) be any sequence of unitary braid reps. Then TFAE:

- ρ_n is localizable, with R finite order
- $|\rho_n(B_n)| < \infty$
- eigenvalues λ_n of inclusion matrices $G^{(n)}$ satisfy $\lambda_n^L \in \mathbb{Z}$.

Consequently

A localizable topological quantum computer cannot be universal.

Thank You!