Localizing Topological Quantum Computers

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Outline

First Definitions

Definition

Topological Quantum Computation is a computational model built upon systems in topological phases.

Definition (Nayak, et al '08)

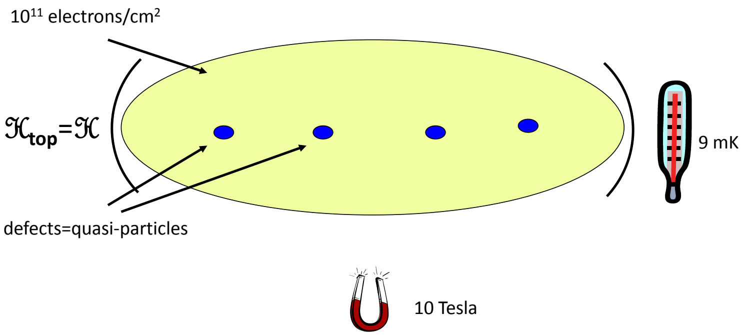
a system is in a topological phase if its low-energy effective field theory is a topological quantum field theory (TQFT).

Fact

Most (known, useful) $(2+1)$-TQFTs come from the Reshetikhin-Turaev construction via modular tensor categories (MTCs).

Example: FQH Liquid Cartoon

Fractional Quantum Hall Liquid

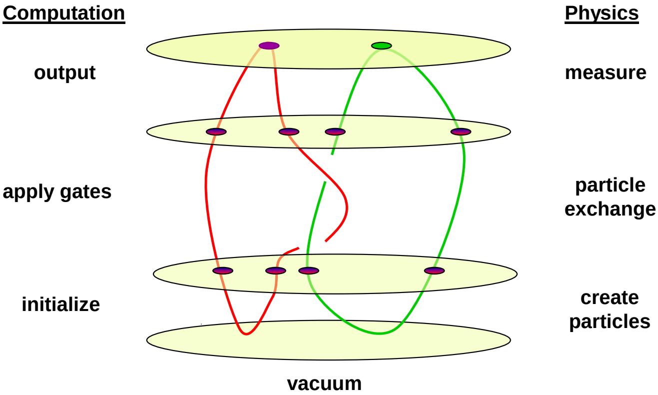


Topological Quantum Computation

Description

- Computational space: $\mathcal{H}(n)$ *n*-particle state space
- Quantum gates: particle exchange
- Measurement: quasi-particles fusion

Topological Model (non-adaptive)



Topological Quantum Field Theory

A (unitary) 3D **TQFT** assigns to any (compact oriented labelled extended) surface (M, ℓ) a (finite-dimensional) Hilbert space: $\mathcal{H}(M, \ell)$ satisfying some compatibility axioms.

Remarks

Label set $\mathcal{L}$:

- Mathematically: simple objects in a modular category $\mathcal{C}$
  $\mathbf{1} = X_0$ is unit object.
- Physically: indecomposable quasi-particle types $\mathbf{1}$ labels vacuum.

Basic Data

$\mathcal{H}(M,\ell)$ determined by axioms and value of $\mathcal{H}$ on basic pieces:

Remarks

Basic pieces are:

- empty: $\mathcal{H}(\emptyset)=\mathbb{C}$
- disk: $\mathcal{H}(D^2;i)=\begin{cases}\mathbb{C} & i=0\\ 0 & else\end{cases}$
- annulus: $\mathcal{H}(A;a,b)=\begin{cases}\mathbb{C} & a=\hat{b}\\ 0 & else\end{cases}$
- pants: $P:=D^2\setminus\{z_1,z_2\}$ $\mathcal{H}(P;a,b,c)=\mathbb{C}^{N(a,b,c)}$
  $N(a,b,c)$ are fusion rules of MTC $\mathcal{C}$.

Key axioms

Axiom (Disjoint Sum)

$\mathcal{H}((M_1,\ell_1)\sqcup(M_2,\ell_2)) = \mathcal{H}(M_1,\ell_1)\otimes\mathcal{H}(M_2,\ell_2)$

Axiom (Gluing)

If M_g is obtained from gluing two boundary circles of M together then

$$\mathcal{H}(M_g,\ell) = \bigoplus_{x\in\mathcal{L}} \mathcal{H}(M,(\ell,x,\hat{x}))$$

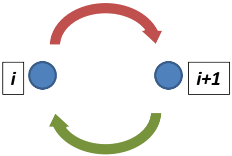
The Braid Group

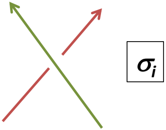
Definition

$\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$





Topological Model: Algebraic Description

Remarks

- $\mathcal{H}(n) \subset \mathrm{End}(X^{\otimes n})$, $X \in \mathcal{C}$ MTC
- Gate set $S = \{\varphi_n(\sigma_i)\}$ where

$$\varphi_n : \mathcal{B}_n \to \mathbf{U}(\mathrm{End}(X^{\otimes n}) \twoheadrightarrow \mathbf{U}(\mathcal{H}(n))$$

- output: (approximations of) link polynomial evaluations

Quantum Circuit Model vs. TQC

- State space a tensor power: $\dim(\mathbb{C}^{d})^{\otimes n} = d^n$
- Gates are local: act on $n_i \ll n$ qu-dits.
- decoherence

- State space a direct sum: $\dim \mathcal{H}(n) \neq c^n$
- Gates are global: $\varphi_n(\sigma_i)$ *smeared* across $\mathcal{H}(n)$
- Fault-tolerant

Main Question

Question

Is there a topological model that is both:

- universal and
- local(izable)?

Sequences of Braid Representations

Let $\iota : \mathcal{B}_n \to B_{n+1}$, $\iota(\sigma_i) = \sigma_i$ for $i \leq n-1$.

Definition

A **sequence of braid representations** is a family of $\mathcal{B}_n$-reps (ρ_n, V_n) and maps τ_n such that the following diagram commutes for all n:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & \longrightarrow & \mathbb{C}\rho_n(\mathcal{B}_n) \\ \downarrow \iota & & \downarrow \tau_n \\ \mathbb{C}\mathcal{B}_{n+1} & \longrightarrow & \mathbb{C}\rho_{n+1}(\mathcal{B}_{n+1}) \end{array}$$

Braided Vector Spaces

Definition

(R, V) is a **braided vector space** if $R \in Aut(V \otimes V)$ satisfies

$$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R)$$

Induces a sequence of $\mathcal{B}_n$-reps $(\rho_R, V^{\otimes n})$ where

$$\rho_R(\sigma_i) = I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}$$

Representations are local: $\rho_n(\sigma_i)$ acts on tensor factors $i, i+1$.

Example

Example

Let $V = \mathbb{C}^2$ and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $\rho_R : \mathcal{B}_n \to \mathbf{U}(V^{\otimes n})$.

Notice: $|\rho_R(\mathcal{B}_n)| < \infty$: finite group.

Jones Representations

Definition

$\mathcal{C}(\mathfrak{sl}_2, \ell)$ is the MTC associated with $U_q\mathfrak{sl}_2$ with $q = e^{\pi i/\ell}$.

$$\mathbb{C}(q)\mathcal{B}_n \twoheadrightarrow \mathrm{End}(X_1) \cong \bigoplus_i \mathrm{End}(V_i^{(n)})$$

induces unitary **Jones representations**:

Definition

$\rho_n^{(\ell)} : \mathcal{B}_n \rightarrow \prod_i \mathbf{U}(V_i^{(n)})$

Denote by $(\rho_n^{(\ell)}, W_n^{(\ell)})$ with $W_n^{(\ell)} = \bigoplus_i V_i^{(n)}$

Contact with FQH Liquids

FQH Liquids distinguished by **filling fraction** $\nu = p/q$.

Conjecture (Read-Rezayi)

The TQFTs modelling FQH liquids at $\nu = 2 + \frac{k}{k+2}$ is the $SU(2)$ level k Chern-Simons TQFT (i.e. Jones reps at $\ell = k + 2$)

Examples

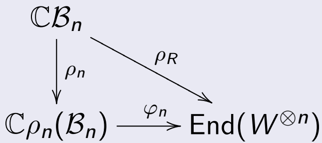
ν	TQFT	Description
1/3	$SU(2)_1$	Abelian
5/2	$SU(2)_2$	Ising (non-universal)
13/5	$SU(2)_3$	Fibonacci (universal)
8/3	$SU(2)_4$	non-universal

Formal Definition

Definition

Suppose (ρ_n, V_n) is a sequence of completely reducible braid representations. A **localization** of (ρ_n, V_n) is a braided vector space (W, R) such that for all $n \geq 2$:

(i) there exist $\varphi_n : \mathbb{C}\rho_n(\mathcal{B}_n) \to \mathrm{End}(W^{\otimes n})$ such that the following diagram commutes for all n:



(ii) and $(\varphi_n, W^{\otimes n})$ is a faithful $\mathbb{C}\rho_n(\mathcal{B}_n)$-module.

In other words...

If (R, W) localizes (ρ_n, V_n),

- Decompose (ρ_n, V_n): $V_n \cong \bigoplus_{i \in J_n} V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- then $W^{\otimes n} \cong \bigoplus_{i \in J_n} \mu_n^i V_n^{(i)}$ as a $\mathbb{C}\mathcal{B}_n$-module
- with $\mu_n^i > 0$ (strictly positive multiplicities)

Remarks

- $\dim(V_n) \neq d^n$ (usually), so extra copies of some $V_n^{(i)}$ needed.
- (R, W) uniformly localizes for all $n \geq 2$.
- $\overrightarrow{\mu}_n$ **localization vector**.

Motivating Example: $\ell = 4$

Let $R = \alpha \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ with $\alpha = -\frac{e^{-\pi i/4}}{\sqrt{2}}$.

Theorem (Franko,R,Wang '06)

$$(\mathbb{C}^2)^{\otimes n} \cong \begin{cases} (\sqrt{2})^n V_n^{(1)} \oplus (\sqrt{2})^n V_n^{(2)} & n \quad even \\ (\sqrt{2})^{n+1} V_n^{(1)} & n \quad odd \end{cases}$$

First Result

Theorem (R,Wang)

Unitary Jones reps. $(\rho_n^{(\ell)}, \bigoplus_i V_i^{(n)})$ *localizable if, and only if* $\ell \in \{3,4,6\}$

Compare with:

Theorem (Freedman,Larsen,Wang 2001)

$\{\rho_n^{(\ell)}(\sigma_i)\}$ *is a universal gate set for QC if, and only if* $\ell \notin \{3,4,6\}$ *(take* $n \gg 0$*).*

Bratteli Diagrams

Consider irreducible $\mathcal{B}_n$-rep $V_i^{(n)}$.

How does $V_i^{(n)}|_{\mathcal{B}_{n-1}}$ decompose?

$$V_i^{(n)} \cong \bigoplus_j m_{ij}^{(n-1)} V_j^{(n-1)}$$

Recorded in **Inclusion Matrix** $G^{(n-1)} := [m_{ij}^{(n-1)}]_{ij}$ or

Bratteli diagram

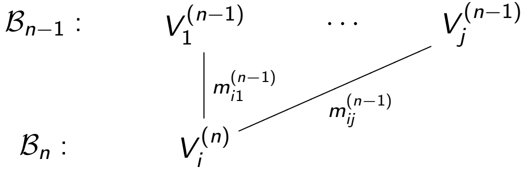
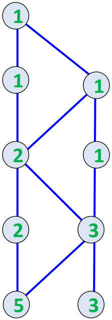
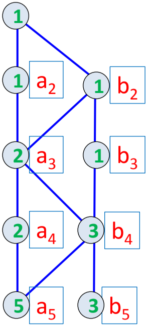


Illustration: Fibonacci ($\ell = 5$)



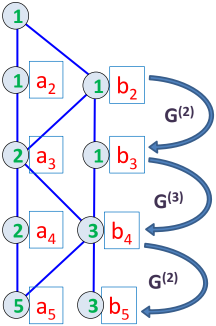
If (R, V) localizes $\rho_n^{(5)}$

Illustration: Fibonacci ($\ell = 5$)



If (R, V) localizes $\rho_n^{(5)}$ with mult. vectors (a_n, b_n)

Illustration: Fibonacci ($\ell = 5$)



If (R, V) localizes $\rho_n^{(5)}$ with mult. vectors (a_n, b_n) then by Perron-Frobenius Theorem

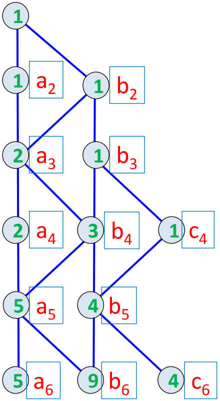
$$G^{(3)} G^{(2)} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \lambda \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

where $G^{(3)} G^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\lambda = \left(\frac{1 \pm \sqrt{5}}{2}\right)^2, \ a_2, b_2 \in \mathbb{Z}.$$

Impossible!

Example: $\ell = 6$



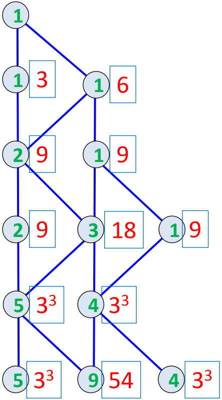
If (R, V) localizes $\rho_n^{(6)}$ with $\dim(V) = k$ then

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \lambda \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

and $2a_3 + b_3 = k^3$

$k = \lambda = 3$, $a_3 = b_3 = 9$ works!

Example: $\ell = 6$



Is there a 9×9 R-matrix?

$$\gamma \begin{pmatrix} \omega & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega \\ 0 & \omega & 0 & 0 & 0 & \omega & 1 & 0 & 0 \\ 0 & 0 & \omega & \omega^2 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 & \omega & 0 & 0 & 0 & \omega^2 & 0 \\ \omega & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \omega & \omega & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 1 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & \omega^2 & 0 & 0 & 0 & \omega & 0 \\ 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & \omega \end{pmatrix}$$

Localizes $\rho_n^{(6)}$.

YBE Conjecture

Conjecture (R,Wang)

Suppose $R \in \mathrm{End}(V \otimes V)$ solution to (YBE) is:

- Unitary
- finite order ($R^k = I$)

Then $\rho_R(\mathcal{B}_n)$ is finite for all n.

Complexity of Jones Polynomial Evaluations

Classically we have:

Theorem (Vertigan,Wocjan-Yard)

Exact computation of $J_L(q)$ at $q = e^{2\pi i/\ell}$ is:

$$\begin{cases} FP & \ell = 3,4,6 \\ FP^{\#P} & else \end{cases}$$

And on a quantum computer:

Theorem

Approximation of $J_L(q)$ at $q = e^{2\pi i/\ell}$ is BQP-complete for $\ell \neq 3, 4, 6$.

Localization Conjecture

Conjecture (R,Wang)

Let (ρ_n, V_n) be any sequence of unitary braid reps. Then TFAE:

- ρ_n is localizable, with R finite order
- $|\rho_n(B_n)| < \infty$
- FP-eigenvalues λ_n of inclusion matrices $G^{(n)}$ satisfy $\lambda_n^L \in \mathbb{Z}$.
- Associated link invariants efficiently computable on Turing Machine.

Conclusion

A localizable topological quantum computer cannot be universal (using braiding alone).

Outlook

Remarks

- *Property* **F** *Conjecture* [R, '07]: Let $\mathcal{C}$ be a braided fusion category. Then $|\rho_X(\mathcal{B}_n)| < \infty$ for all n if and only if $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$. $\mathrm{FPdim}(X)$ is FP-eigenvalue of $G^{(n)}$
- For X with $\mathbb{C}[\rho_X(\mathcal{B}_n)] \cong \mathrm{End}(X^{\otimes n})$ localizable $\Rightarrow$ $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ (arXiv:1009.0241).
- May need generalized or quasi- localizations.

Conclusion

Non-universality and localizability are detected by $\mathrm{FPdim}(X)$.

Thank You!