Outline

1. Modular Categories
   - Recollection
   - Fusion Rules and Dimensions

2. Problem I
   - Enumerative Classification
   - Techniques
   - Algorithms and Examples

3. Problem II
   - Braid Group Images
   - Conceptual Evidence
   - Techniques
   - Empirical Evidence
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Some Axioms

Definition

A modular category is a monoidal category \((\mathcal{C}, \otimes, 1)\) that is: \(\mathbb{C}\)-linear, abelian, finite rank, semisimple, rigid, braided, balanced, non-degenerate.
Grothendieck Semiring

Definition

\[ Gr(C) := (\text{Obj}(C), \oplus, \otimes, 1) \text{ a unital based ring.} \]

Representation \( \varphi : Gr(C) \to \text{Mat}_m(\mathbb{Z}) \)
defined by: \( \varphi(X_i) = N_i \)
where \( (N_i)_{k,j} := \dim \text{Hom}(X_i \otimes X_j, X_k) \)
Frobenius-Perron Dimensions

Definition

- \( \text{FPdim}(X) \) is the largest eigenvalue of \( \varphi(X) \)
- \( \text{FPdim}(\mathcal{C}) := \sum_{i=0}^{m-1} \text{FPdim}(X_i)^2 \)

\( \text{FPdim} : \text{Gr}(\mathcal{C}) \rightarrow \mathbb{C} \) is unique positive character.
If \( \text{FPdim}(X) = \dim(X) \) for all \( X, \mathcal{C} \) is pseudo-unitary.
Integrality

**Definition**

\( C \) is

- **integral** if \( \text{FPdim}(X) \in \mathbb{Z} \) for all \( X \)
- **weakly integral** if \( \text{FPdim}(C) \in \mathbb{Z} \)

[Etingof,Nikshych,Ostrik ’05]: \( C \) integral iff \( \cong \text{Rep}(H) \), \( H \) f.d. s.s. quasi-Hopf alg.
Sources of Modular Categories

Example
- Quantum groups type: $\mathcal{C}(\mathfrak{g}, q, \ell)$
- $\text{Rep}(D^\omega G)$, $G$ a finite group, $\omega$ a 3-cocyle
- Drinfeld centers: $Z(C)$ for $C$ a spherical category.
Wang’s Conjecture

Conjecture

Fix $m \geq 1$. There are finitely many modular categories of rank $m$. 
Reduction

**Theorem (Ocneanu Rigidity)**

Fix a unital based ring $R$. There are at most finitely many (fusion, ribbon) modular categories $\mathcal{C}$ with $\text{Gr}(\mathcal{C}) \cong R$.

Up to finite ambiguity, enough consider:

**Problem**

Classify all unital based rings $R$ such that $R \cong \text{Gr}(\mathcal{C})$ for some modular category $\mathcal{C}$.

**Definition**

$\mathcal{C}$ and $\mathcal{D}$ are Grothendieck equivalent if $\text{Gr}(\mathcal{C}) \cong \text{Gr}(\mathcal{D})$. 
Modest Goal

Problem

Classify modular categories:
- that are pseudo-unitary (so \( \text{dim}(X_i) \geq 1 \))
- up to Grothendieck equivalence
- for small ranks \( m \) (say \( \leq 12 \)).
Symmetries of $N_{i,j}^k$

Denote by $i^*$ the label of $X_i^*$. Then:

- $N_{i,j}^k = N_{j,i}^k = N_{i,k^*}^{i^*} = N_{i^*,j^*}^k$
- $N_{i,i^*}^0 = 1$, $N_i^{i^*} = N_i^T$
- $N_i N_j = N_j N_i$

Diophantine equations.
Symmetries of $S$ and $T$

Define $T_{i,j} = \theta_i \delta_{i,j}$. $S$ and $T$ satisfy:

- $S = S^T$ (symmetric, but $\mathbb{C}$-entries!)
- $SS^\dagger = \alpha I$, $S^4 = \alpha^2 I$ (Projectively Unitary)
- $T^N = I$ for some $N$, $\theta_i^* = \theta_i$.
- $(ST)^3 = \gamma S^2$
- $\dim(C) = (\sum_i \theta_i \dim(X_i)^2)(\sum_i \dim(X_i)^2 / \theta_i)$

Note: \[
\begin{pmatrix}
0 & -1 \\
1 & 0 \\
\end{pmatrix} \rightarrow S \quad \text{and} \quad \begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix} \rightarrow T \quad \text{give rep. of } PSL(2, \mathbb{Z}).\]
Verlinde formula:

\[ N_{i,j}^k = \sum_r \frac{S_{i,r} S_{j,r} S_{k,r}}{\dim(C) \dim(X_r)}. \]

So enough to classify $S$-matrices...
Moreover, $\Phi_j(X_i) := S_{i,j} / \dim(X_j)$ are characters of $Gr(C)$. 
The balancing axiom:

\[ S_{i,j} \theta_i \theta_j = \sum_k N_{i*,j}^k \dim(X_k) \theta_k. \]

**Definition**

A triple \( \{N_i\}, S, T \) satisfying symmetries, balancing, Verlinde, is a **modular data**.
Galois Theory

Preliminary Facts:

- \( S_{i,j} \) are algebraic integers and \( \theta_i \) are roots of 1.
- Denote \( p_i(x) := \text{char}_{N_i}(x) \).
- Roots of \( p_i \) are \( S_{i,j} / \dim(X_j) \)
- \( K = \mathbb{Q}(\{S_{i,j} : 0 \leq i, j \leq m - 1\}) \) is Galois extension
- \( K \) is splitting field of \( \{p_i\} \).
Denote by $G = \text{Gal}(K/\mathbb{Q}) = \text{Gal}(C)$.

**Theorem (Coste, Gannon)**

(a) $G$ is abelian

(b) $G$ permutes characters $\Phi_j$, get injective homomorphism: $G \rightarrow S_m$.

(c) Let $\sigma \in G < S_m$: $S_{i,j} = \pm S_{\sigma(i),\sigma^{-1}(j)}$
More Galois Theory

**Proposition (Ng, Schauenburg)**

\[ S_{i,j} \in \mathbb{Q}(\{\theta_k : 1 \leq k \leq m - 1\}). \]

**Conjecture (Sommerhäuser)**

\[ [\mathbb{Q}(\{\theta_i\}_i) : K] \text{ divides } 8. \]
Fix \( m \) and \( G \leq S_m \) abelian. Classify modular categories \( \mathcal{C} \) with \( \text{Gal}(\mathcal{C}) = G \) (not just \( \cong \)).

Remark

- \( j \) is \( G \)-fixed if and only if \( S_{i,j}/\dim(X_j) \in \mathbb{Z} \) for all \( i \).
- \( 0 \) is \( G \)-fixed if and only if \( \mathcal{C} \) is integral.
- If \( G \) does not fix any \( j \), lots of symmetries among the \( S_{i,j} \).
Lemma

Suppose $\mathcal{C}$ is an integral modular category of rank $m$. Let $d_1 \geq d_2 \geq \cdots \geq d_m = 1$ be the dimensions of simple objects. Then the integers $y_i := \dim(\mathcal{C})/(d_i)^2$ satisfy:

(a) $y_1 \leq y_2 \leq \cdots \leq y_m = \dim(\mathcal{C})$

(b) $\sum_{i=1}^{m} 1/y_i = 1$

(c) $i \leq y_i \leq (m - i + 1)u_i$ where $u_1 := 1$ and $u_{k+1} := u_k(u_k + 1)$.

$u_k + 1$ is Sylvester’s sequence: $2, 3, 7, 43, 1807, \ldots$ (double-exponential).
To classify integral modular categories of rank $m$:

1. Solve for integers $y_1 \leq \cdots \leq y_m$ such that:
   1. $\sqrt{y_m/y_i} \in \mathbb{Z}$
   2. $\sum_i 1/y_i = 1$
   3. $x_i \leq u_m$

2. Set $d_i = \sqrt{y_m/y_i}$. solve $d_id_j = \sum_k N_{i,j}^kd_k$, $N_{i,j}^k \leq \max(d_i)^2$.

3. solve for $S$ and $T$, look for realizations...
Example: \( m = 5 \)

- so \( y_i \leq 1807 \)
- Only solutions: \((5, 5, 5, 5, 5)\) and \((2, 8, 8, 8, 8)\).
- Gives dimensions: \((1, 1, 1, 1, 1)\) and \((1, 1, 1, 1, 2)\)
- \((1, 1, 1, 1, 1)\) is realized, \((1, 1, 1, 1, 2)\) is not (no \( S \) matrix).
Non-integral Strategy

Fix $m$ and $G < S_m$. Assume $|G(0)| > 1$ (otherwise integral)

- Use $G$-action and $SS^\dagger \propto I$
- compute $\text{char}_{X_i}(x)$ two ways: from $N_i$ and roots: $S_{i,j}/\text{dim}(X_j)$.
- gives more Diophantine equations in $N_{i,j}^k$.
- Use balancing to bound $M > [\mathbb{Q}(\theta_i) : \mathbb{Q}] = [\mathbb{Q}(\theta_i) : K][K : \mathbb{Q}]$.
- So $\varphi_n(\theta_i) = 0$, $\varphi_n$ cyclotomic degree $n < M$.
- Usually, implies Diophantine eqn in each $N_{i,j}^k$ (finitely many solutions)
A rank 3 example

Let \( S = \begin{pmatrix} 1 & d_1 & d_2 \\ d_1 & s_{1,1} & s_{1,2} \\ d_2 & s_{1,2} & s_{2,2} \end{pmatrix} \), suppose \( G = \langle (012) \rangle \)

\( G \)-symmetries imply:
\[ s_{1,1} = \pm d_2, \quad s_{1,2} = \pm 1 \quad \text{and} \quad s_{2,2} = \pm d_1 \quad (\pm \text{ independent}). \]

\( S \) unitary and \( d_i > 1 \) (not integral) imply:
\[ S = \begin{pmatrix} 1 & d_1 & \frac{d_1}{d_1-1} \\ d_1 & -\frac{d_1}{d_1-1} & 1 \\ \frac{d_1}{d_1-1} & 1 & -d_1 \end{pmatrix}. \]

Observe: \( \text{det}(N_1) = \text{det}(N_2) = -1 \).

e.g. \( d_1\left(\frac{-1}{d_1-1}\right)\left(\frac{d_1-1}{d_1}\right) = -1 \)
example, continued

\[ N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & n_1 & n_2 \\ 0 & n_2 & n_3 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & n_2 & n_3 \\ 1 & n_3 & n_4 \end{pmatrix} \]

\[ N_1 N_2 = N_2 N_1 \quad \text{and} \quad \det(N_1) = \det(N_2) = -1 \]

imply:

\[ n_2 = n_3 = 1 \quad \text{and} \quad n_1 + n_4 = 1, \quad \text{so up to} \quad X_1 \leftrightarrow X_2, \quad n_1 = 1, \quad n_4 = 0. \]

This is realized (subcategory of \( C(\mathfrak{sl}_2, 7) \)).
Non-self-dual approach

Complex conjugation $c = (1 \ 1^*) \cdots (i \ i^*) \cdots \in G$
$c \neq 1$ if $X_i \not\cong X_i^*$ for some $i$.
$G$ abelian, so this gives significant reduction. For example:

Theorem

Suppose $C$ is an odd rank, maximally non-self-dual modular category (i.e. only $1$ is self-dual simple). Then $C$ is integral.

Proof.

After relabeling, $c = (1 \ 2) \cdots (m - 2 \ m - 1) \in G$. No $\sigma \in G$ with $\sigma(0) \neq 0$ commutes with $c$. So $G(0) = \{0\}$, hence integral.
Techniques lead to classification of (pseudo-unitary) modular categories if:

1. rank $\leq 4$
2. non-self-dual, rank $\leq 5$
3. integral, rank $\leq 6$
**Fact**

*Braiding on $\mathcal{C}$ induces:*

\[
\Psi_X : \mathbb{C}B_n \to \text{End}(X^{\otimes n})
\]

\[
\sigma_i \to \text{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \text{Id}_X^{\otimes n-i-1}
\]
Property \( F \)

**Definition**

Say \( \mathcal{C} \) has **property \( F \)** if \( |\Psi_X(\mathcal{B}_n)| < \infty \) for all \( X \) and \( n \).

- \( \cdots \subset \Psi_X(\mathcal{B}_n) \subset \Psi_X(\mathcal{B}_{n+1}) \subset \cdots \)
  so if no property \( F \), \( |\Psi_X(\mathcal{B}_n)| = \infty \) for all \( n \gg 0 \)

- If \( Y \subset X \otimes^k \) then \( \Psi_X(\mathcal{B}_{kn}) \rightarrow \Psi_Y(\mathcal{B}_n) \)
  so to verify prop. \( F \), check for generating \( X \).
A braided fusion category $\mathcal{C}$ has property F if and only if it is weakly integral ($\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$).
Verifying property $F$

$C$ is group-theoretical if

- $\mathcal{Z}(C) \cong \text{Rep}(D^\omega G)$ [Natale ’03], or
- $\mathcal{Z}(C) \cong \mathcal{Z}(P)$, $P$ a pointed category.

Proposition (Etingof, R., Witherspoon ’08)

Braided group-theoretical categories $C$ have property $F$.

Proof.

Braided functor $C \hookrightarrow \mathcal{Z}(C) \cong \text{Rep}(D^\omega G)$.

Reduces to $\text{Rep}(D^\omega G)$.

$\mathcal{B}_n$ acts on $(D^\omega G)^\otimes n$ as monomial group.
Weakly Group Theoretical Categories

Definition

- $\mathcal{D}$ is nilpotent if $\mathcal{D}_{ad} \supset (\mathcal{D}_{ad})_{ad} \supset \cdots$ converges to $\text{Vec}$.
- $\mathcal{C}$ is weakly group theoretical if $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}\mathcal{D}$ for $\mathcal{D}$ nilpotent.

- $\mathcal{C}$ weakly group theoretical $\Rightarrow$ $\mathcal{C}$ weakly integral
- Conjecturally, $\Leftarrow$, so
- Enough to show weakly group theoretical $\mathcal{C}$ have property $F$?
Failure of Property $F$ (Philosophy)

- Enough to show $B_3$ has infinite image on $\text{Hom}(Y, X \otimes^3)$. 
- If $G \subset GL(V)$ finite, irreducible, primitive, $|g|$, $g \in G$ small (relative to $\dim(V)$).
- $\text{FPdim}(C) \not\in \mathbb{Z} \iff [\mathbb{Q}(\{\theta_i\}; i) : \mathbb{Q}]$ large.
- Eigenvalues of $g_i = \Psi_X(\sigma_i)$ products of $\sqrt{\theta_i}$ hence
- $|g_i|$ too large, so $|G| = \infty$. 

Two approaches:

1. If \( C \) is integral, try to show \( C \) is group theoretical (see [Naidu,R]).

2. If \( C(q, q, \ell) \) is weakly integral related to \( TL_n(q) \), \( \text{Hecke}_n(q) \) or \( \text{BMW}_n(q, r) \), determine \( \Psi_X(B_n) \) explicitly.

3. Otherwise, good luck! (New techniques needed).
Proposition (R, Tuba)

Suppose $(\rho, V)$ is an irreducible $\mathbb{C}$-representation of $B_3$ with $\dim(V) \leq 5$ and $\rho(B_3)$ finite and primitive. Then, projectively, $|\rho(\sigma_1)| \leq 24$.

Remark

- Imprimitive $G$ also understood.
- More refined result available—nearly always have sufficient conditions for $|\rho(B_3)| < \infty$. 
An Example

Proposition

Property F conjecture is true for $C(g_2, \ell)$.

Proof.

(outline) Let $X$ be “7-dimensional” object, assume $3 \mid \ell$.

1. For $\ell \geq 18$, $\dim \text{Hom}(X^3, X) = 4$ and $B_3$ acts irreducibly.
2. $\text{Spec}(\Psi_X(\sigma_1)) = \{q^{-12}, q^2, -q^{-6}, -1\}$.
3. $|\Psi_X(B_3)| = \infty$ for $0 << \ell$ (use [R,Tuba])
4. Check $FPdim(X)^2 \not\in \mathbb{Z}$. Verify for small $\ell$. 
Proposition (Preliminary)

*Non-weakly-integral* $\mathcal{C}(g, q, \ell)$ fails to have property $F$.

Property $F$ verified for *weakly integral* $\mathcal{C}(g, q, \ell)$ except for non-integral $\mathfrak{so}_{2k+1}$ at $\ell = 4k + 2$ and $\mathfrak{so}_{2m}$ at $\ell = m$ (i.e. when $\sqrt{2k + 1}, \sqrt{m} \not\in \mathbb{Z}$).
Lie types $B$ and $D$

**Conjecture**

- $C(\mathfrak{so}_{2k+1}, 4k + 2)$ has property $F$
- $C(\mathfrak{so}_{2m}, 2m)$ has property $F$

- Difficulty: spin objects $V_\epsilon$. Description of $\Psi_{V_\epsilon}(\mathbb{C}B_n)$?
- $\text{FPdim}(V_\epsilon) \in \{\sqrt{2k + 1}, \sqrt{m}\}$
- Verified for $k \leq 4, m \leq 5$
- Property $F$ fails otherwise [Larsen, R, Wang ’05].
$\mathcal{P} = \mathcal{C}(s_0 p, 2p), \ p$ prime

set $X := V_\epsilon$

simples:

$\{1, Z, X, X', Y_1, \ldots, Y_k\}$

$\text{FPdim}(X) = \text{FPdim}(X') = \sqrt{p}$

$\text{FPdim}(Y_i) = 2, \ \text{FPdim}(Z) = 1$

$\text{dim Hom}(X^\otimes n, X) = \frac{p^{n-1}}{2} + 1$

$\text{dim Hom}(X^\otimes n, X') = \frac{p^{n-1}}{2} - 1$
Guesses?

Look for a series of finite (simple) groups with irreps of dimensions:

\[ p \frac{n-1}{2} + 1 \] and \[ p \frac{n-1}{2} - 1 \]

Any guesses?

Conjecture

PSp(2n, p) (Weil representation.)

This has been verified for \( p = 3, 5 \) and 7
Let $R \in M_{m^2}(\mathbb{C})$ be a unitary solution to:

$$R_1 R_2 R_1 = R_2 R_1 R_2$$

where $R_1 = (R \otimes I)$ and $R_2 = (I \otimes R)$ and $R$ has finite order.

**Question**

Image of $\mathcal{B}_n \to U(\mathbb{C}^{mn})$ finite?

**Results**

- If $R$ comes from $D^\omega G$: Yes.
- For $m = 2$: Yes [Franko, R, Wang '05], [Franko, Thesis].
Conversely...

\[ \Psi_X : \mathbb{C}B_n \rightarrow \text{End}(X \otimes^n) \] “non-local” while for \( X \in \text{Rep}(D^\omega G) \) \( B_n \) acts \textit{locally}: on \( X \otimes^n \).

**Fact**

Integrality of \( C \) not necessary: \( C(\mathfrak{sl}_2, 4) \) and \( C(\mathfrak{sl}_2, 6) \) \( B_n \) reps. can be unitarily localized.

**Question (Wang)**

Unitarily localized iff weakly integral?
Thank you!