Modular Categories and Applications II

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Outline

Some Axioms

Definition

A modular category is a monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ that is: $\mathbb{C}$-linear, abelian, finite rank, semisimple, rigid, braided, balanced, non-degenerate.

Grothendieck Semiring

Definition

$Gr(\mathcal{C}) := (Obj(\mathcal{C}), \oplus, \otimes, \mathbf{1})$ a unital based ring.

Representation $\varphi : Gr(\mathcal{C}) \to \mathrm{Mat}_m(\mathbb{Z})$
defined by: $\varphi(X_i) = N_i$
where $(N_i)_{k,j} := \dim \mathrm{Hom}(X_i \otimes X_j, X_k)$

Frobenius-Perron Dimensions

Definition

- FPdim(X) is the largest eigenvalue of $\varphi(X)$
- $\mathrm{FPdim}(\mathcal{C}) := \sum_{i=0}^{m-1} \mathrm{FPdim}(X_i)^2$

$\mathrm{FPdim} : Gr(\mathcal{C}) \to \mathbb{C}$ is unique positive character.
If $\mathrm{FPdim}(X) = \dim(X)$ for all X, $\mathcal{C}$ is pseudo-unitary.

Integrality

Definition

$\mathcal{C}$ is

- integral if $\mathrm{FPdim}(X) \in \mathbb{Z}$ for all X
- weakly integral if $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$

[Etingof,Nikshych,Ostrik '05]: $\mathcal{C}$ integral iff $\mathcal{C} \cong \mathrm{Rep}(H)$, H f.d. s.s. quasi-Hopf alg.

Sources of Modular Categories

Example

- Quantum groups type: $\mathcal{C}(\mathfrak{g}, q, \ell)$
- $\mathrm{Rep}(D^{\omega}G)$, G a finite group, ω a 3-cocyle
- Drinfeld centers: $Z(\mathcal{C})$ for $\mathcal{C}$ a spherical category.

Wang's Conjecture

Conjecture

Fix $m \geq 1$. There are finitely many modular categories of rank m.

Reduction

Theorem (Ocneanu Rigidity)

Fix a unital based ring R. There are at most finitely many (fusion, ribbon) modular categories $\mathcal{C}$ with $Gr(\mathcal{C}) \cong R$.

Up to finite ambiguity, enough consider:

Problem

Classify all unital based rings R such that $R \cong Gr(\mathcal{C})$ for some modular category $\mathcal{C}$.

Definition

$\mathcal{C}$ and $\mathcal{D}$ are Grothendieck equivalent if $Gr(\mathcal{C}) \cong Gr(\mathcal{D})$.

Modest Goal

Problem

Classify modular categories:

- that are pseudo-unitary (so $\dim(X_i) \geq 1$)
- up to Grothendieck equivalence
- for small ranks m (say ≤ 12).

Symmetries of $N_{i,j}^k$

Denote by i^* the label of X_i^*. Then:

- $N_{i,j}^k = N_{j,i}^k = N_{i,k^*}^{j^*} = N_{i^*,j^*}^{k^*}$
- $N_{i,i^*}^0 = 1$, $N_{i^*} = N_i^T$
- $N_i N_j = N_j N_i$

Diophantine equations.

Symmetries of S and T

Define $T_{i,j} = \theta_i \delta_{i,j}$. S and T satisfy:

- $S = S^T$ (symmetric, but $\mathbb{C}$-entries!)
- $SS^\dagger = \alpha I$, $S^4 = \alpha^2 I$ (Projectively Unitary)
- $T^N = I$ for some N, $\theta_{i^*} = \theta_i$.
- $(ST)^3 = \gamma S^2$
- $\dim(\mathcal{C}) = (\sum_i \theta_i \dim(X_i)^2)(\sum_i \dim(X_i)^2/\theta_i)$

Note: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow S$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow T$ give rep. of $PSL(2, \mathbb{Z})$.

Relationship between N_i and S

Verlinde formula:

$$N_{i,j}^{k} = \sum_r \frac{S_{i,r} S_{j,r} \overline{S_{k,r}}}{\dim(\mathcal{C}) \dim(X_r)}.$$

So enough to classify S-matrices...
Moreover, $\Phi_j(X_i) := S_{i,j} / \dim(X_j)$ are characters of $Gr(\mathcal{C})$.

Relationship Among N_i, S and T

The balancing axiom:

$$S_{i,j}\theta_i\theta_j = \sum_k N_{i^*,j}^k \dim(X_k)\theta_k.$$

Definition

A triple $(\{N_i\}, S, T)$ satisfying symmetries, balancing, Verlinde, is a modular data.

Galois Theory

Preliminary Facts:

- $S_{i,j}$ are algebraic integers and θ_i are roots of 1.
- Denote $p_i(x) := char_{N_i}(x)$.
- roots of p_i are $S_{i,j}/\dim(X_j)$
- $K = \mathbb{Q}(\{S_{i,j} : 0 \leq i,j \leq m-1\})$ is Galois extension
- K is splitting field of $\{p_i\}$.

Harder Results

Denote by $G = Gal(K/\mathbb{Q}) = Gal(\mathcal{C})$.

Theorem (Coste, Gannon)

(a) *G is abelian*

(b) *G permutes characters Φ_j, get* injective *homomorphism: $G \to S_m$.*

(c) *Let $\sigma \in G < S_m$: $S_{i,j} = \pm S_{\sigma(i),\sigma^{-1}(j)}$*

More Galois Theory

Proposition (Ng,Schauenburg)

$S_{i,j} \in \mathbb{Q}(\{\theta_k : 1 \leq k \leq m-1\})$.

Conjecture (Sommerhäuser)

$[\mathbb{Q}(\{\theta_i\}_i) : K]$ divides 8.

New Strategy

Problem

Fix m and $G < S_m$ abelian. Classify modular categories $\mathcal{C}$ with $Gal(\mathcal{C}) = G$ (not just $\cong$).

Remark

- j is G-fixed if and only if $S_{i,j}/\dim(X_j) \in \mathbb{Z}$ for all i.
- 0 is G-fixed if and only if $\mathcal{C}$ is integral.
- If G does not fix any j, lots of symmetries among the $S_{i,j}$.

Integral Cases

Lemma

Suppose $\mathcal{C}$ is an integral modular category of rank m. Let $d_1 \geq d_2 \geq \cdots \geq d_m = 1$ be the dimensions of simple objects. Then the integers $y_i := \dim(\mathcal{C})/(d_i)^2$ satisfy:

(a) $y_1 \leq y_2 \leq \cdots \leq y_m = \dim(\mathcal{C})$

(b) $\sum_{i=1}^{m} 1/y_i = 1$

(c) $i \leq y_i \leq (m - i + 1)u_i$ *where* $u_1 := 1$ *and* $u_{k+1} := u_k(u_k + 1)$.

$u_k + 1$ is Sylvester's sequence: $2, 3, 7, 43, 1807, \ldots$ (double-exponential).

Algorithm

To classify integral modular categories of rank m:

1. Solve for integers $y_1 \leq \cdots \leq y_m$ such that:
    1. $\sqrt{y_m/y_i} \in \mathbb{Z}$
    2. $\sum_i 1/y_i = 1$
    3. $x_i \leq u_m$
2. Set $d_i = \sqrt{y_m/y_i}$. solve $d_i d_j = \sum_k N_{i,j}^k d_k$, $N_{i,j}^k \leq \max(d_i)^2$.
3. solve for S and T, look for realizations...

Example: $m = 5$

- so $y_i \leq 1807$
- Only solutions: $(5, 5, 5, 5, 5)$ and $(2, 8, 8, 8, 8)$.
- Gives dimensions: $(1, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 2)$
- $(1, 1, 1, 1, 1)$ is realized, $(1, 1, 1, 1, 2)$ is not (no S matrix).

Non-integral Strategy

Fix m and $G < S_m$. Assume $|G(0)| > 1$ (otherwise integral)

- Use G-action and $SS^\dagger \propto I$
- compute $char_{X_i}(x)$ two ways: from N_i and roots: $S_{i,j}/\dim(X_j)$.
- gives more Diophantine equations in $N_{i,j}^k$.
- Use balancing to bound $M > [\mathbb{Q}(\theta_i) : \mathbb{Q}] = [\mathbb{Q}(\theta_i) : K][K : \mathbb{Q}]$.
- So $\varphi_n(\theta_i) = 0$, φ_n cyclotomic degree $n < M$.
- Usually, implies Diophantine eqn *in each* $N_{i,j}^k$ (finitely many solutions)

A rank 3 example

Let $S = \begin{pmatrix} 1 & d_1 & d_2 \\ d_1 & s_{1,1} & s_{1,2} \\ d_2 & s_{1,2} & s_{2,2} \end{pmatrix}$, suppose $G = \langle (012) \rangle$

G-symmetries imply:

$s_{1,1} = \pm d_2$, $s_{1,2} = \pm 1$ and $s_{2,2} = \pm d_1$ ($\pm$ independent).

S unitary and $d_i > 1$ (not integral) imply:

$$S = \begin{pmatrix} 1 & d_1 & \frac{d_1}{d_1-1} \\ d_1 & \frac{-d_1}{d_1-1} & 1 \\ \frac{d_1}{d_1-1} & 1 & -d_1 \end{pmatrix}.$$

Observe: implies $\det(N_1) = \det(N_2) = -1$.

e.g. $d_1(\frac{-1}{d_1-1})(\frac{d_1-1}{d_1}) = -1$

example, continued

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & n_1 & n_2 \\ 0 & n_2 & n_3 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & n_2 & n_3 \\ 1 & n_3 & n_4 \end{pmatrix}$$

$N_1 N_2 = N_2 N_1$ and $\det(N_1) = \det(N_2) = -1$ imply:
$n_2 = n_3 = 1$ and $n_1 + n_4 = 1$, so up to $X_1 \leftrightarrow X_2$, $n_1 = 1$, $n_4 = 0$.
This is realized (subcategory of $\mathcal{C}(\mathfrak{sl}_2, 7)$).

Non-self-dual approach

Complex conjugation $c = (1\ 1^*) \cdots (i\ i^*) \cdots \in G$
$c \neq 1$ if $X_i \ncong X_{i^*}$ for some i.
G abelian, so this gives significant reduction. For example:

Theorem

Suppose $\mathcal{C}$ is an odd rank, maximally non-self-dual modular category (i.e. only $\mathbf{1}$ is self-dual simple). Then $\mathcal{C}$ is integral.

Proof.

After relabeling, $c = (1\ 2) \cdots (m-2\ m-1) \in G$. No $\sigma \in G$ with $\sigma(0) \neq 0$ commutes with c. So $G(0) = \{0\}$, hence integral. □

Results

Techniques lead to classification of (pseudo-unitary) modular categories if:

Results

1. rank ≤ 4
2. non-self-dual, rank ≤ 5
3. integral, rank ≤ 6

Notation

Fact

Braiding on $\mathcal{C}$ induces:

$$\Psi_X : \mathbb{C}\mathcal{B}_n \rightarrow \mathrm{End}(X^{\otimes n})$$

$$\sigma_i \rightarrow \mathrm{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{Id}_X^{\otimes n-i-1}$$

Property F

Definition

Say $\mathcal{C}$ has property **F** if $|\Psi_X(\mathcal{B}_n)| < \infty$ for all X and n.

- $\cdots \subset \Psi_X(\mathcal{B}_n) \subset \Psi_X(\mathcal{B}_{n+1}) \subset \cdots$
  so if no property **F**, $|\Psi_X(\mathcal{B}_n)| = \infty$ for all $n >> 0$
- If $Y \subset X^{\otimes k}$ then $\Psi_X(\mathcal{B}_{kn}) \twoheadrightarrow \Psi_Y(\mathcal{B}_n)$
  so to verify prop. **F**, check for generating X.

Property **F** Conjecture

Conjecture

A braided fusion category $\mathcal{C}$ has property **F** if and only if it is weakly integral ($\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$).

Verifying property F

$\mathcal{C}$ is group-theoretical if

- $\mathcal{Z}(\mathcal{C}) \cong \mathrm{Rep}(D^\omega G)$ [Natale '03], or
- $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{P})$, $\mathcal{P}$ a *pointed* category.

Proposition (Etingof,R.,Witherspoon '08)

Braided group-theoretical categories $\mathcal{C}$ have property **F**.

Proof.

Braided functor $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C}) \cong \mathrm{Rep}(D^\omega G)$.
Reduces to $\mathrm{Rep}(D^\omega G)$.
$\mathcal{B}_n$ acts on $(D^\omega G)^{\otimes n}$ as monomial group. □

Weakly Group Theoretical Categories

Definition

- $\mathcal{D}$ is nilpotent if $\mathcal{D}_{ad} \supset (\mathcal{D}_{ad})_{ad} \supset \cdots$ converges to Vec.
- $\mathcal{C}$ is weakly group theoretical if $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}\mathcal{D}$ for $\mathcal{D}$ *nilpotent*.

- $\mathcal{C}$ weakly group theoretical $\Rightarrow$ $\mathcal{C}$ weakly integral
- Conjecturally, $\Leftarrow$, so
- Enough to show weakly group theoretical $\mathcal{C}$ have property F?

Failure of Property F (Philosophy)

- Enough to show $\mathcal{B}_3$ has infinite image on $\mathrm{Hom}(Y, X^{\otimes 3})$.
- If $G \subset GL(V)$ finite, irreducible, primitive, $|g|$, $g \in G$ small (relative to $\dim(V)$).
- $\mathrm{FPdim}(\mathcal{C}) \notin \mathbb{Z} \leftrightsquigarrow [\mathbb{Q}(\{\theta_i\}_i) : \mathbb{Q}]$ large.
- Eigenvalues of $g_i = \Psi_X(\sigma_i)$ products of $\sqrt{\theta_i}$ hence
- $|g_i|$ too large, so $|G| = \infty$.

Verifying Property F in Practice

Two approaches:

1. If $\mathcal{C}$ is integral, try to show $\mathcal{C}$ is group theoretical (see [Naidu,R]).
2. If $\mathcal{C}(\mathfrak{g}, q, \ell)$ is weakly integral related to $TL_n(q)$, Hecke$_n(q)$ or BMW$_n(q, r)$, determine $\Psi_X(\mathcal{B}_n)$ explicitly.
3. Otherwise, good luck! (New techniques needed).

Failure of Property F in Practice

Proposition (R,Tuba)

Suppose (ρ, V) *is an irreducible* $\mathbb{C}$*-representation of* $\mathcal{B}_3$ *with* $\dim(V) \leq 5$ *and* $\rho(\mathcal{B}_3)$ *finite and primitive. Then, projectively,* $|\rho(\sigma_1)| \leq 24$.

Remark

- Imprimitive G also understood.
- More refined result available–nearly always have sufficient conditions for $|\rho(\mathcal{B}_3)| < \infty$.

An Example

Proposition

Property **F** *conjecture is true for* $\mathcal{C}(\mathfrak{g}_2, \ell)$.

Proof.

(outline) Let X be "7-dimensional" object, assume $3 \mid \ell$.

1. For $\ell \geq 18$, $\dim \mathrm{Hom}(X^3, X) = 4$ and $\mathcal{B}_3$ acts irreducibly.
2. $Spec(\Psi_X(\sigma_1))$: $\{q^{-12}, q^2, -q^{-6}, -1\}$.
3. $|\Psi_X(\mathcal{B}_3)| = \infty$ for $0 << \ell$ (use [R,Tuba])
4. Check $\mathrm{FPdim}(X)^2 \notin \mathbb{Z}$. Verify for small ℓ.

□

Quantum Group types

Proposition (Preliminary)

Non-weakly-integral $\mathcal{C}(\mathfrak{g}, q, \ell)$ *fails to have property* F.

Property F verified for weakly integral $\mathcal{C}(\mathfrak{g}, q, \ell)$ except for non-integral $\mathfrak{so}_{2k+1}$ at $\ell = 4k+2$ and $\mathfrak{so}_{2m}$ at $\ell = m$ (i.e. when $\sqrt{2k+1}, \sqrt{m} \notin \mathbb{Z}$).

Lie types B and D

Conjecture

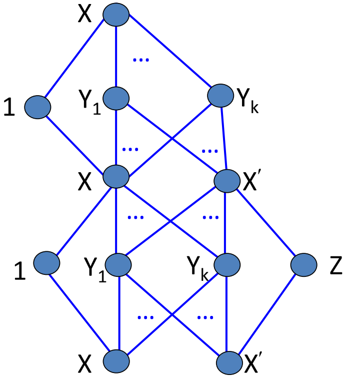
- $\mathcal{C}(\mathfrak{so}_{2k+1}, 4k+2)$ has property **F**
- $\mathcal{C}(\mathfrak{so}_{2m}, 2m)$ has property **F**

- Difficulty: spin objects V_ϵ. Description of $\Psi_{V_\epsilon}(\mathbb{C}\mathcal{B}_n)$?
- $\mathrm{FPdim}(V_\epsilon) \in \{\sqrt{2k+1}, \sqrt{m}\}$
- Verified for $k \leq 4$, $m \leq 5$
- Property **F** fails otherwise [Larsen,R,Wang '05].

Some Details

$\mathcal{P} = \mathcal{C}(\mathfrak{so}_p, 2p)$, p prime
set $X := V_\epsilon$
simples:
$\{\mathbf{1}, Z, X, X', Y_1, \ldots, Y_k\}$
$\mathrm{FPdim}(X) = \mathrm{FPdim}(X') = \sqrt{p}$
$\mathrm{FPdim}(Y_i) = 2$, $\mathrm{FPdim}(Z) = 1$
$\dim \mathrm{Hom}(X^{\otimes n}, X) = \frac{p^{\frac{n-1}{2}}+1}{2}$
$\dim \mathrm{Hom}(X^{\otimes n}, X') = \frac{p^{\frac{n-1}{2}}-1}{2}$

Bratteli Diagram



Guesses?

Look for a series of finite (simple) groups with irreps of dimensions:

$$\frac{p^{\frac{n-1}{2}}+1}{2} \text{ and } \frac{p^{\frac{n-1}{2}}-1}{2}$$

Any guesses?

Conjecture

PSp$(2n, p)$ (Weil representation.)

This has been verified for $p = 3, 5$ and 7

Local Braidings

Let $R \in M_{m^2}(\mathbb{C})$ be a unitary solution to:
$R_1 R_2 R_1 = R_2 R_1 R_2$ where $R_1 = (R \otimes I)$ and $R_2 = (I \otimes R)$ and R has finite order.

Question

Image of $\mathcal{B}_n \to U(\mathbb{C}^{mn})$ finite?

Results

- If R comes from $D^\omega G$: Yes.
- For $m = 2$: Yes [Franko,R,Wang '05], [Franko, Thesis].

Conversely...

$\Psi_X : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$ "non-local" while for $X \in \mathrm{Rep}(D^\omega G)$ $\mathcal{B}_n$ acts locally: on $X^{\otimes n}$.

Fact

Integrality of $\mathcal{C}$ not necessary: $\mathcal{C}(\mathfrak{sl}_2, 4)$ and $\mathcal{C}(\mathfrak{sl}_2, 6)$ $\mathcal{B}_n$ reps. can be unitarily localized.

Question (Wang)

Unitarily localized iff weakly integral?

Thank you!