Braided Weakly Integral Fusion Categories

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Outline

Some Axioms

Definition

A fusion category $\mathcal{C}$ is a monoidal category that is:

- $\mathbb{C}$-linear, abelian
- finite rank: simple classes $\{X_0 := \mathbf{1}, X_1, \ldots, X_{m-1}\}$
- semisimple
- rigid: duals X^*, $b_X : \mathbf{1} \to X \otimes X^*$, $d_X : X^* \otimes X \to \mathbf{1}$
- compatibility...

Braiding

Definition

A braided fusion (BF) category has (a natural family of) isomorphisms:

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying, e.g.

$$c_{X,Y\otimes Z} \quad = \quad (\mathrm{Id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{Id}_Z)$$

Sources of Braided Fusion Categories

Example

- semisimple subquotient $\mathcal{C}(\mathfrak{g}, \ell)$ of $\text{Rep}(U_q\mathfrak{g})$ at $q = e^{\pi i/\ell}$ is BF cat.
- twisted double of finite group G: $D^\omega G$ $\text{Rep}(D^\omega G)$ is BF cat.
- $\mathcal{C}$ a fusion cat. then Drinfeld center $\mathcal{Z}(\mathcal{C})$ is BF cat.

Grothendieck Semiring

Definition

$Gr(\mathcal{C}) := (Obj(\mathcal{C}), \oplus, \otimes, \mathbf{1})$ a unital based ring.

- Define matrices $(N_i)_{k,j} := \dim \mathrm{Hom}(X_i \otimes X_j, X_k)$
- Rep. $\varphi : Gr(\mathcal{C}) \rightarrow \mathrm{End}(\mathbb{Z}^m)$

$$\varphi(X_i) = N_i$$

- Respects duals: $\varphi(X^*) = \varphi(X)^T$ (self-dual $\Rightarrow$ symmetric)
- If $\mathcal{C}$ is braided, $Gr(\mathcal{C})$ is commutative

Frobenius-Perron Dimensions

Definition

- FPdim(X) is the largest eigenvalue of $\varphi(X)$
- $\mathrm{FPdim}(\mathcal{C}) := \sum_{i=0}^{m-1} \mathrm{FPdim}(X_i)^2$

(a) $\mathrm{FPdim}(X) > 0$

(b) $\mathrm{FPdim} : Gr(\mathcal{C}) \to \mathbb{C}$ is a unital homomorphism

(c) FPdim is unique with (a) and (b).

(Weak) Integrality

Definition

$\mathcal{C}$ is

- integral if $\mathrm{FPdim}(X) \in \mathbb{Z}$ for all X
- weakly integral if $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$

[Etingof,Nikshych,Ostrik '05]: $\mathcal{C}$ integral iff $\mathcal{C} \cong \mathrm{Rep}(H)$, H f.d. s.s. quasi-Hopf alg.

Equivalently...

Lemma

$\mathcal{C}$ weakly integral iff $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ for all simple X.

Subfactor $N \subset M$ associated with
$\cdots \subset \mathrm{End}(X^{\otimes n}) \subset \mathrm{End}(X^{\otimes n+1}) \subset \cdots$ has $[M : N] = \mathrm{FPdim}(X)^2$.
So weakly integral fusion category $\leftrightarrow$ integer Jones index.

The Braid Group

Definition

$\mathcal{B}_n$ has generators σ_i, $i = 1, \ldots, n-1$ satisfying:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1$$

Braid Group Representations

Fact

Braiding on $\mathcal{C}$ induces:

$$\Psi_X : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$$

$$\sigma_i \to \mathrm{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{Id}_X^{\otimes n-i-1}$$

- X is *not* always a vector space
- $\mathrm{End}(X^{\otimes n})$ semisimple algebra (multi-matrix).
- simple $\mathrm{End}(X^{\otimes n})$-mods $V_k = \mathrm{Hom}(X_k, X^{\otimes n})$ become $\mathcal{B}_n$ reps.
- V_k irred. as $\mathcal{B}_n$ reps. if Ψ_X is surjective.

Property F

Definition

Say $\mathcal{C}$ has property **F** if $|\Psi_X(\mathcal{B}_n)| < \infty$ for all X and n.

- $\cdots \subset \Psi_X(\mathcal{B}_n) \subset \Psi_X(\mathcal{B}_{n+1}) \subset \cdots$
  so if no property **F**, $|\Psi_X(\mathcal{B}_n)| = \infty$ for all $n >> 0$
- If $Y \subset X^{\otimes k}$ then $\Psi_X(\mathcal{B}_{kn}) \twoheadrightarrow \Psi_Y(\mathcal{B}_n)$
  so to verify prop. **F**, check for generating X.

Property **F** Conjecture

Empirical evidence suggests:

Conjecture

A braided fusion category $\mathcal{C}$ has property **F** if and only if it is weakly integral ($\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$).

Quantum Groups of Lie Types A and C

Proposition (Jones '86, Freedman,Larsen,Wang '02)

$\mathcal{C}(\mathfrak{sl}_k, \ell)$ *has property* **F** *if and only if* $\ell \in \{k, k+1, 4, 6\}$.

Proposition (Jones '89, Larsen,R,Wang '05)

$\mathcal{C}(\mathfrak{sp}_{2k}, \ell)$ *has property* **F** *if and only if* $\ell = 10$ *and* $k = 2$.

nly weakly integral in these cases

$$(\mathrm{FPdim}(V) \in \{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, 3\}).$$

Lie types B and D

Conjecture

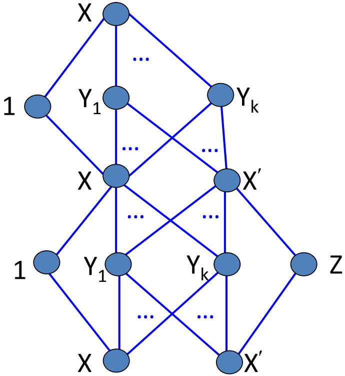
- $\mathcal{C}(\mathfrak{so}_{2k+1}, 4k+2)$ has property **F**
- $\mathcal{C}(\mathfrak{so}_{2m}, 2m)$ has property **F**

- Difficulty: spin objects V_ϵ. Description of $\Psi_{V_\epsilon}(\mathbb{C}\mathcal{B}_n)$?
- $\mathrm{FPdim}(V_\epsilon) \in \{\sqrt{2k+1}, \sqrt{m}\}$
- Verified for $k \leq 4$, $m \leq 5$
- Conjecture true modulo these cases.

Some Details

$\mathcal{P} = \mathcal{C}(\mathfrak{so}_p, 2p)$, p prime
set $X := V_\epsilon$
simples:
$\{\mathbf{1}, Z, X, X', Y_1, \ldots, Y_k\}$
$\mathrm{FPdim}(X) = \mathrm{FPdim}(X') = \sqrt{p}$
$\mathrm{FPdim}(Y_i) = 2$, $\mathrm{FPdim}(Z) = 1$
$\dim \mathrm{Hom}(X^{\otimes n}, X) = \frac{p^{\frac{n-1}{2}}+1}{2}$
$\dim \mathrm{Hom}(X^{\otimes n}, X') = \frac{p^{\frac{n-1}{2}}-1}{2}$

Bratteli Diagram



Useful Tool

$\mathcal{C}$ is group-theoretical if

- $\mathcal{Z}(\mathcal{C}) \cong \text{Rep}(D^\omega G)$ equivalently,
- $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{P})$, $\mathcal{P}$ a *pointed* category.

Proposition (Etingof,R,Witherspoon '08)

Braided group-theoretical categories $\mathcal{C}$ have property **F**.

Proof.

Braided functor $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C}) \cong \text{Rep}(D^\omega G)$.
Reduces to $\text{Rep}(D^\omega G)$.
$\mathcal{B}_n$ acts on $(D^\omega G)^{\otimes n}$ as monomial group. □

Useful Criterion

Proposition (Drinfeld,Gelaki,Nikshych,Ostrik)

An integral modular category $\mathcal{C}$ is group-theoretical if and only if there exists a $\mathcal{D} \subset \mathcal{C}$ such that

- *$\mathcal{D}$ is symmetric and*
- $(\mathcal{D}')_{ad} \subset \mathcal{D}$

- Here $\mathcal{D}'$ is the Müger center:

$$\{X : c_{X,Y} c_{Y,X} = \mathrm{Id}_{X \otimes Y} \, \mathrm{all} \, Y \in \mathcal{D}\}$$

- $\mathcal{L}_{ad}$ is "spanned" by subobjects of all $X \otimes X^*$.

Some Applications

Results (Naidu,R)

- If $\sqrt{2k+1} \in \mathbb{Z}$, $\mathcal{C}(\mathfrak{so}_{2k+1}, 4k+2)$ has property **F**.
- If $\sqrt{m} \in \mathbb{Z}$, $\mathcal{C}(\mathfrak{so}_{2m}, 2m)$ has property **F**.
- If $\mathcal{C}$ a BF category with $\text{FPdim}(X_i) \in \{1,2\}$ and $X^* \cong X$ for all X, $\mathcal{C}$ has property **F**.
- If $\mathcal{C}$ is an integral modular category with $\text{FPdim}(\mathcal{C}) < 36$ or $\text{FPdim}(\mathcal{C}) \in \{pq^2, pq^3\}$, then $\mathcal{C}$ has property **F**.

Weakly Group Theoretical Categories

Definition

- $\mathcal{D}$ is nilpotent if $\mathcal{D}_{ad} \supset (\mathcal{D}_{ad})_{ad} \supset \cdots$ converges to Vec.
- $\mathcal{C}$ is weakly group theoretical if $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{D})$ for $\mathcal{D}$ *nilpotent*.

- $\mathcal{C}$ weakly group theoretical $\Rightarrow$ $\mathcal{C}$ weakly integral
- Conjecturally, $\Leftarrow$, so
- Do weakly group theoretical categories have property **F**?

Braided Vector Spaces

Let $R \in M_{m^2}(\mathbb{C})$ be a unitary solution to:
$R_1 R_2 R_1 = R_2 R_1 R_2$ where $R_1 = (R \otimes I)$ and $R_2 = (I \otimes R)$
Obtain $\mathcal{B}_n$ rep. via $\sigma_i \to R_i = I^{\otimes i-1} \otimes R \otimes I^{\otimes n-i-1}$

Question

Image of $\mathcal{B}_n \to U(\mathbb{C}^{mn})$ finite (projectively)?

Results

- If R comes from $D^\omega G$: Yes.
- For $m = 2$: Yes [Franko,R,Wang '05], [Franko, Thesis].

Conversely...

$\Psi_X : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$ "non-local" while for $X \in \mathrm{Rep}(D^\omega G)$ $\mathcal{B}_n$ acts locally on $X^{\otimes n}$ (X is a v.s.)
However, $\mathcal{C}(\mathfrak{sl}_2, 4)$ only weakly integral and can be "Localized":

Let $R = \alpha \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

$\Psi_X(\mathcal{B}_n) \subset TL_n(i)$ is faithfully represented on $U(\mathbb{C}^{2n})$ via $\sigma_i \to R_i$.

Definition

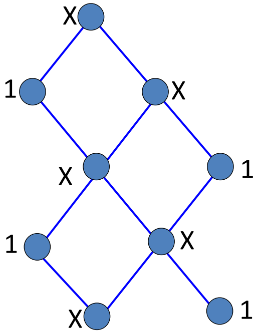
Say Ψ_X can be unitarily localized if there is a unitary R and a v.s. V so that $\Psi_X(\mathcal{B}_n) \to U(V^{\otimes n})$ via $\Psi_X(\sigma_i) \to R_i$ is faithful as a $\Psi_X(\mathbb{C}\mathcal{B}_n)$ rep.

Conjecture (R,Wang)

Unitarily localized iff property **F** (iff weakly integral).

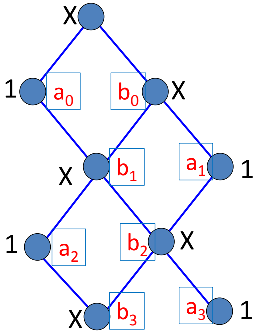
Fibonacci category: $\mathcal{C}(\mathfrak{g}_2, 15)$

Fibonacci Category



Fibonacci category: $\mathcal{C}(\mathfrak{g}_2, 15)$

Fibonacci Category

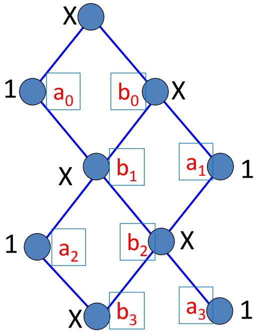


Suppose $R \in M_{m^2}(\mathbb{C})$
a_i copies of $\mathrm{Hom}(X^{\otimes i+2}, \mathbf{1})$
b_i copies of $\mathrm{Hom}(X^{\otimes i+2}, X)$
$m^2 = a_0 + b_0$
$\dim(R[-1]) = a_0$,
$\dim(R[q]) = b_0$
$\dim(R \otimes I[-1]) = ma_0 = b_1$,
$\dim(R \otimes I[q]) = mb_0 = a_1 + b_1$

$$m^k \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

Fibonacci category: $\mathcal{C}(\mathfrak{g}_2, 15)$

Fibonacci Category



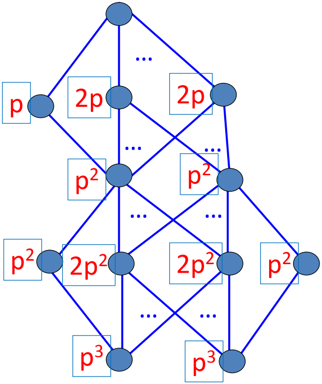
Consequently,

$$\frac{f_{i+2} b_0}{f_{i+1}} \leq m^2 \leq \frac{f_{i+1} b_0}{f_i}$$

hence $\frac{m^2}{b_0} = \frac{1+\sqrt{5}}{2}$
no integer solutions!

$\mathcal{C}(\mathfrak{so}_p, 2p)$ solution

Bratelli Diagram



$R \in M_{p^2}(\mathbb{C})$ could exist:
set $p = 2k + 1$,
$p^2 = p + k(2p)$
$p^n = p^{\frac{n+1}{2}}\left(\frac{p^{\frac{n-1}{2}}+1}{2} + \frac{p^{\frac{n-1}{2}}-1}{2}\right)$
constructed for $p = 3\ldots$

Thank You!