

On the Classification of Modular Tensor Categories

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UT Dallas, 19 Dec. 2007

A Few Collaborators

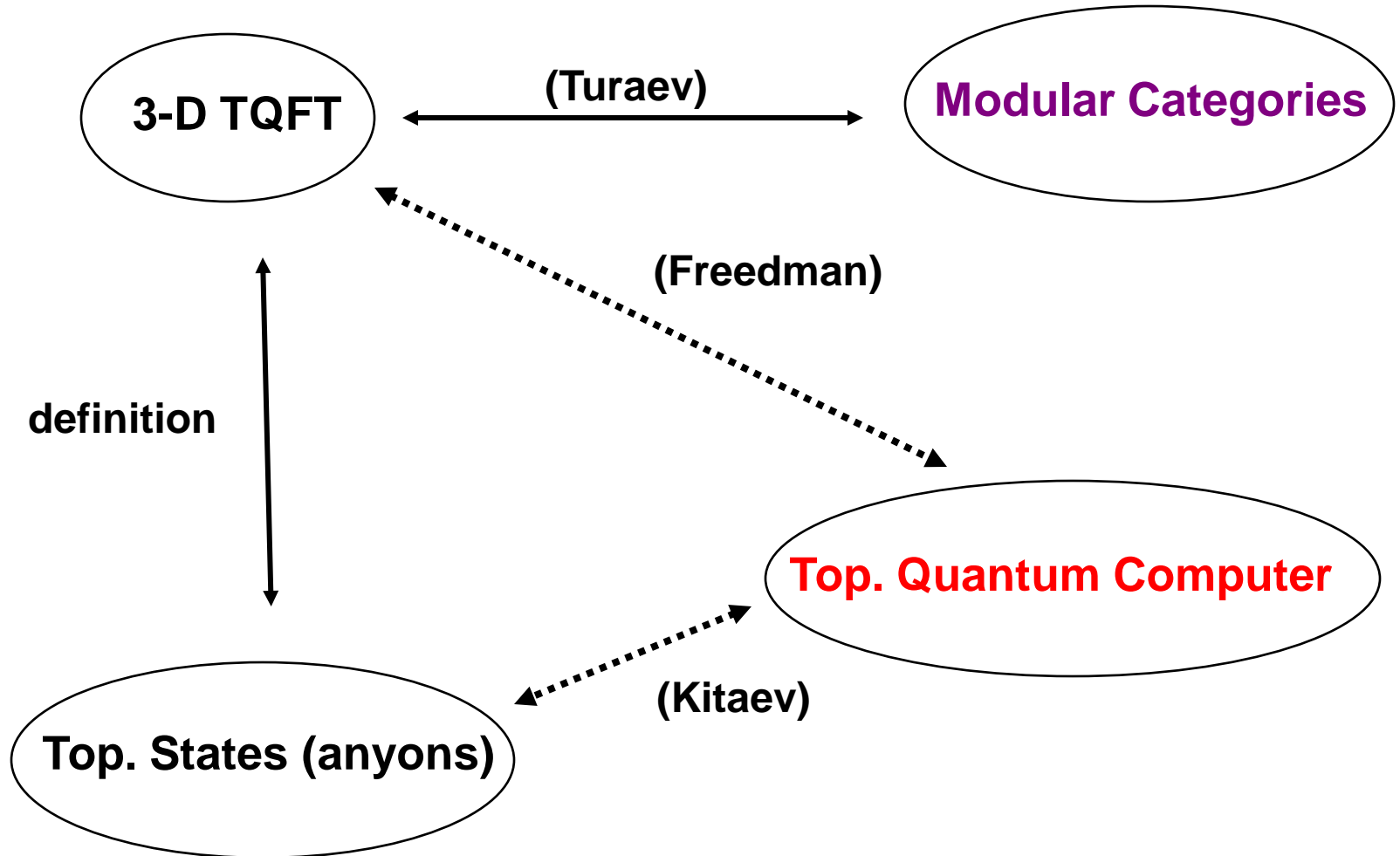
Z. Wang (Microsoft)

R. Stong (Rice)

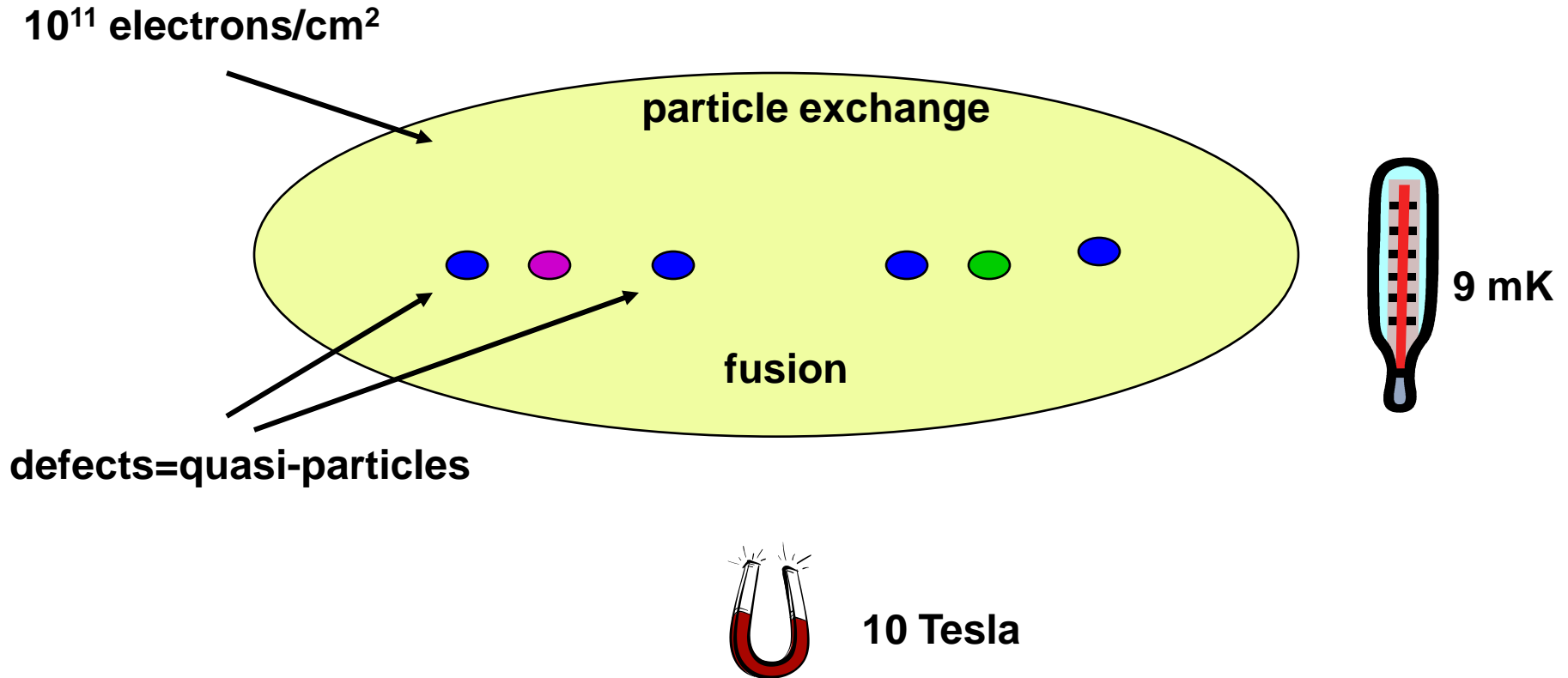
S.-M. Hong (Indiana)

See www.arxiv.org

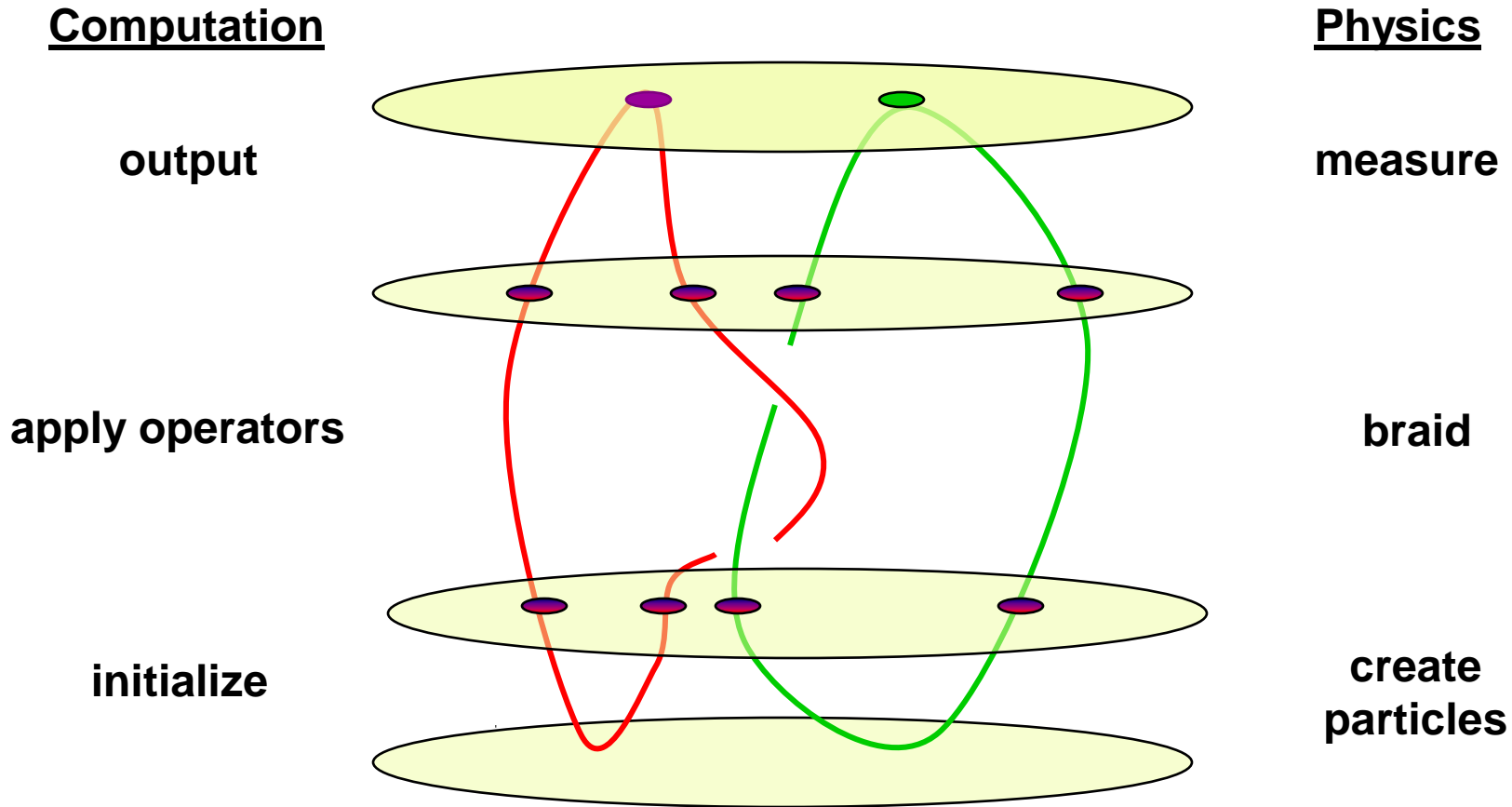
Connections



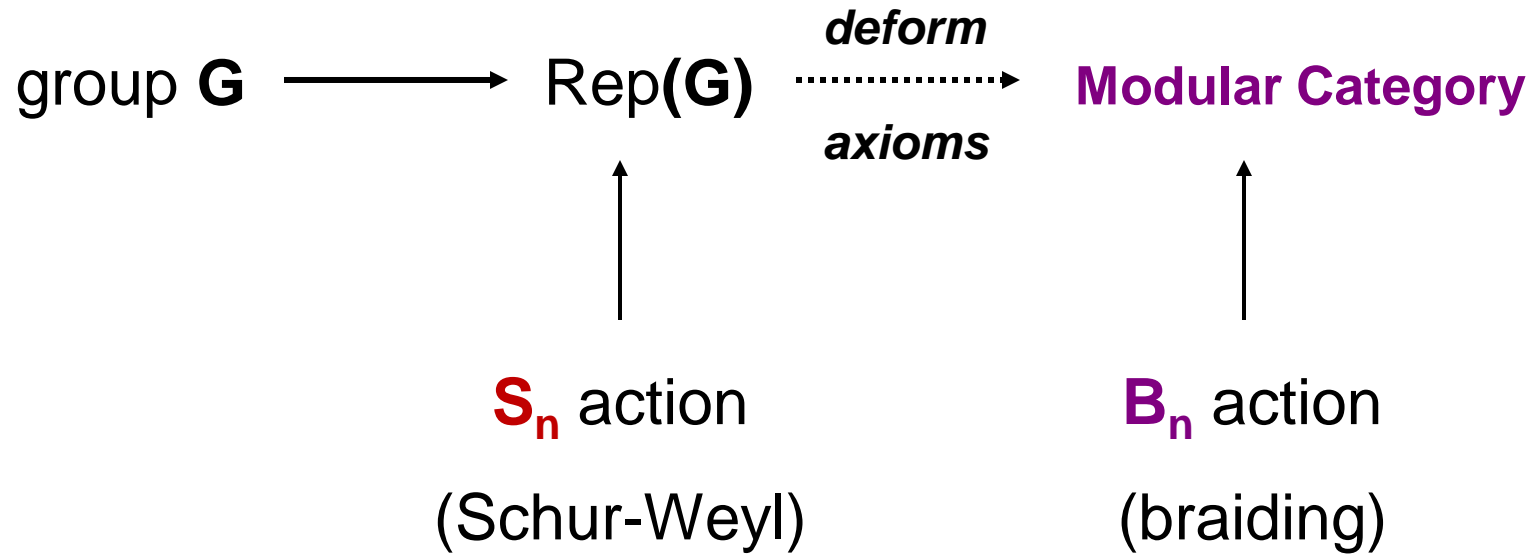
Topological States: FQHE



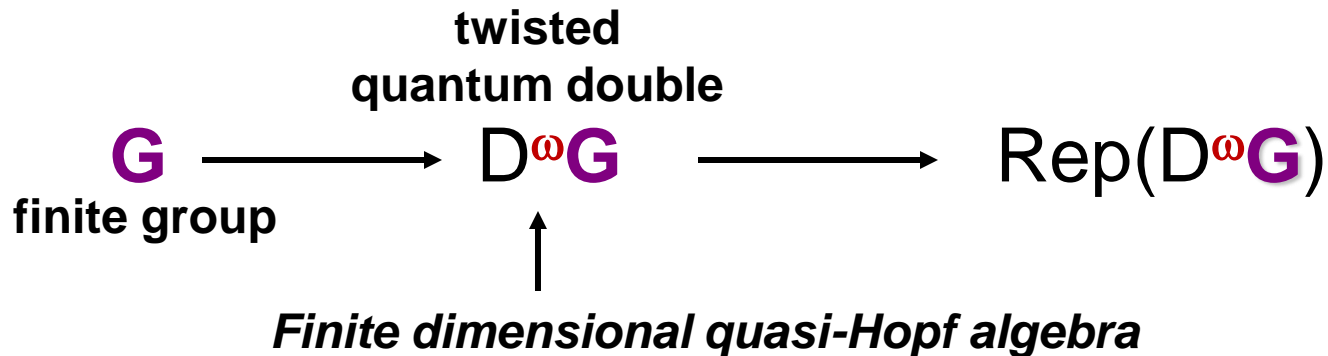
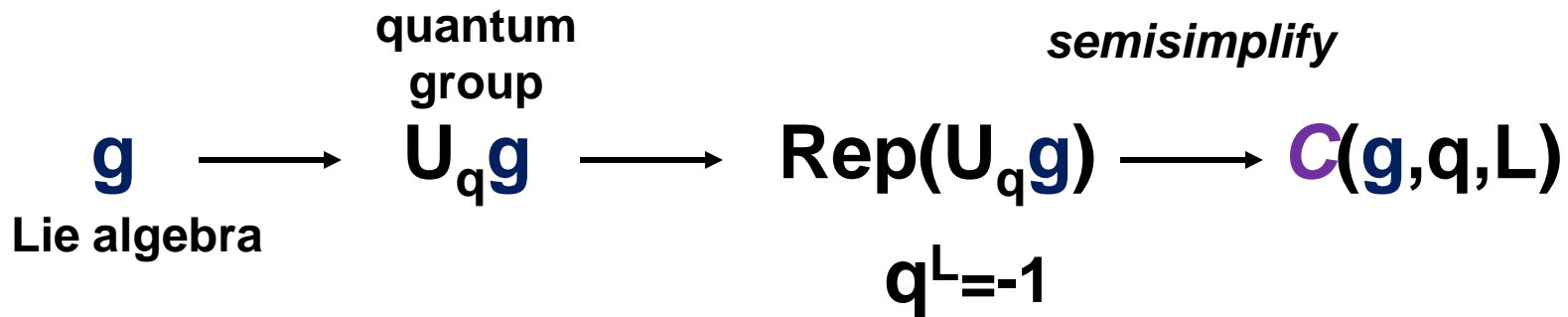
Topological Computation



Conceptual MTC



Algebraic Constructions



New from Old

C and **D** MTCs

- Direct products: $\mathbf{C} \times \mathbf{D}$
- Sub-MTCs: $\mathbf{C} \subset \mathbf{D} \quad (\Rightarrow \mathbf{D} = \mathbf{C} \times \mathbf{C}' \text{ M\"uger})$
- $D(\mathbf{S})$, \mathbf{S} spherical

Invariants of MTCs

- Rank: # of simple objects X_i

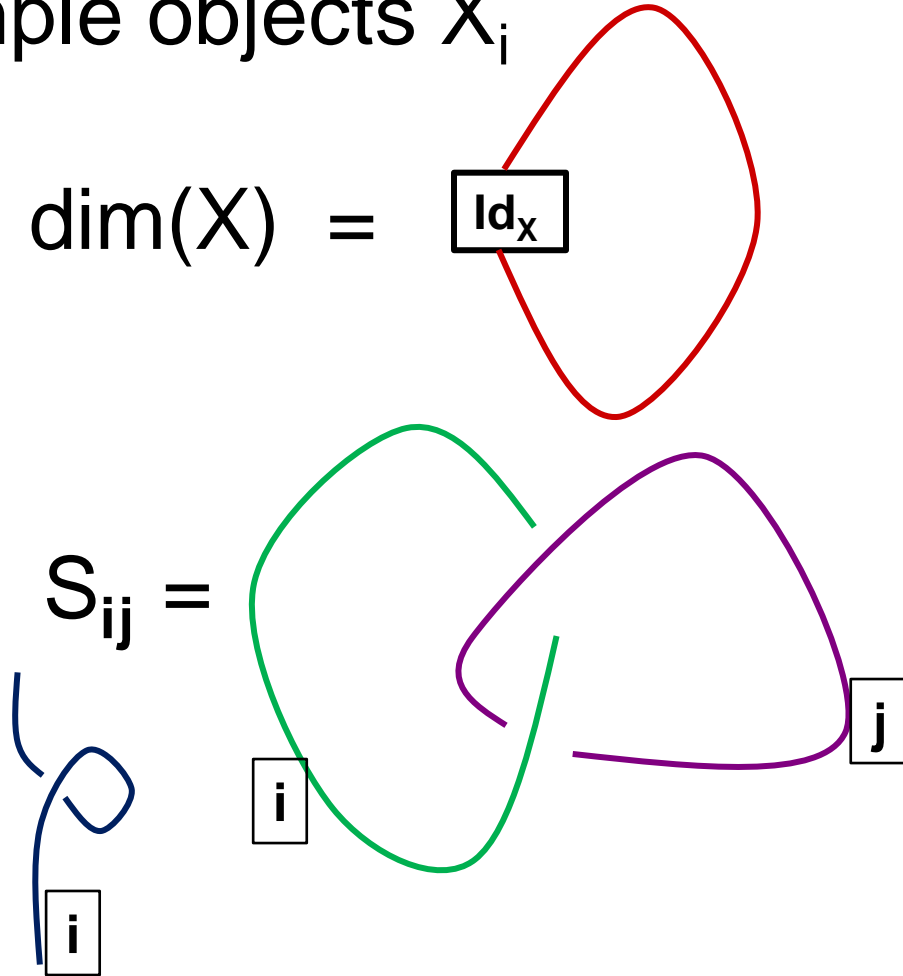
- Dimensions: $\dim(X) = \text{Id}_X$

- S-matrix:

$$S_{ij} =$$

- Twists: $\theta_i =$

$$\theta_i =$$



Structure

- Fusion rules: $\mathbf{X}_i \otimes \mathbf{X}_j = \bigoplus_k N_{ij}^k \mathbf{X}_k$

- \mathbf{B}_n action on $\mathbf{End}(\mathbf{X}^{\otimes n})$:

$$\Phi: \sigma_i \rightarrow \mathbf{Id}_X \otimes \dots \otimes \mathbf{c}_{XX} \otimes \dots \otimes \mathbf{Id}_X$$

- $\mathbf{PSL}(2, \mathbf{Z})$ representation:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \mathbf{S}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \mathbf{T} = \delta_{ij} \theta_i$$

Properties of MTCs

Property	Description
Self-dual	$X \cong X^*$ for all $X \in \mathbf{D}$
Prime	$\mathbf{D} \neq \mathbf{C} \times \mathbf{C}'$
Property F	$\Phi(B_n)$ finite for all $X \in \mathbf{D}$.
Unitary	$\text{End}(X^{\otimes n})$ Hilbert, Φ unitary rep.

Analogy/Counting

N	1	2	3	4	≥ 5
N-dim'l space groups	2	17	230	4783	finite
Rank N UMTCs	2	8	24	36	?

Wang's Conjecture

Conjecture (Z. Wang 2003):

$$\#\{\mathbf{C} \text{ MTC: rank}(\mathbf{C})=\mathbf{M} \} < \infty$$

True for $M \leq 4$!

Another Analogy

Theorem (E. Landau 1903):

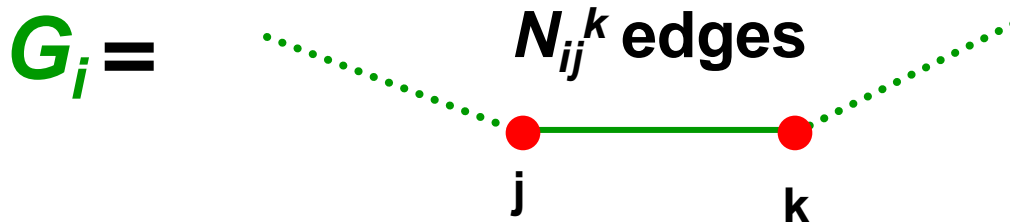
$$\#\{ \mathbf{G} : |\text{Rep}(\mathbf{G})| = N \} < \infty$$

Proof: Exercise 😊 (Hint: Use class equation)

Graphs of Self-dual Fusion Rules

Simple $X_i \rightarrow$ multigraph G_i

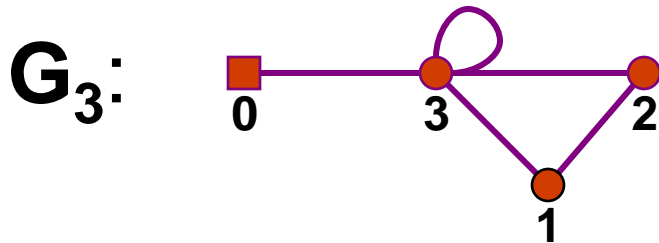
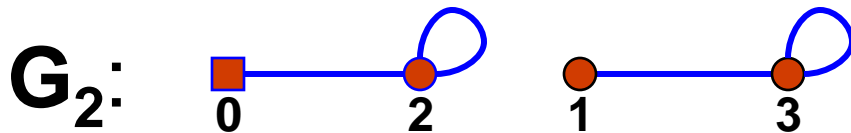
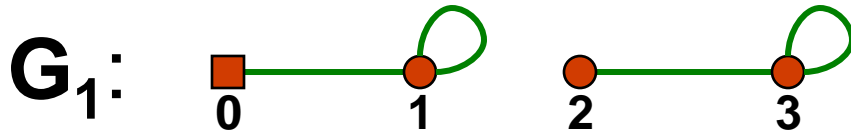
Vertices labeled by $0, \dots, M-1$



Note: for non-self-dual, need arrows...

Example: $C(g_2, q, 10)$

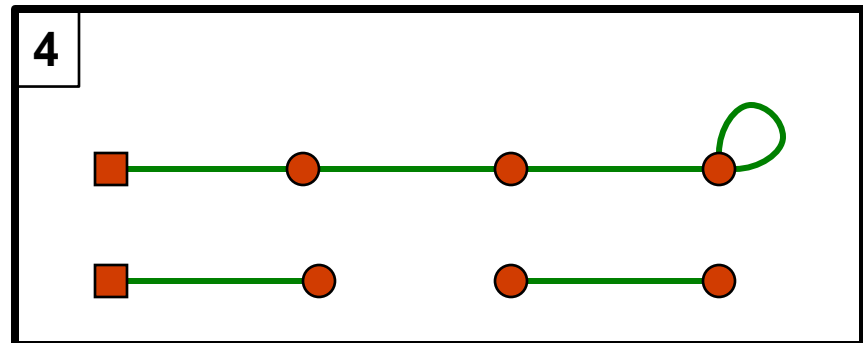
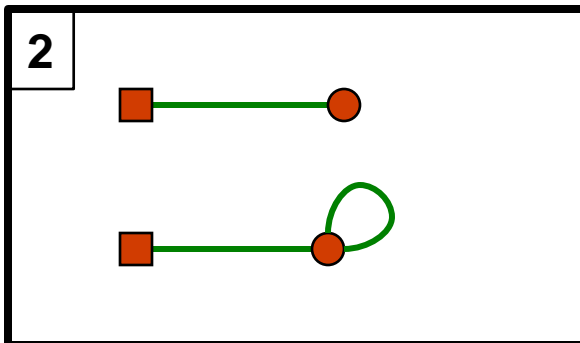
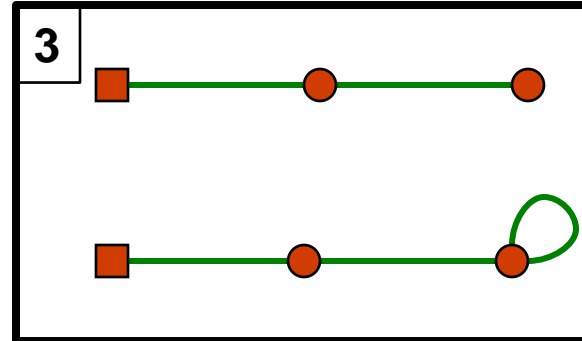
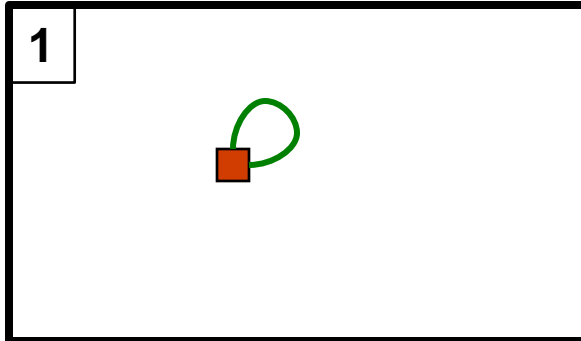
Rank **4** MTC with graphs:



*Not prime, product of
2 copies of **Fibonacci!***

Prime Self-dual UMTCs, rank ≤ 4

Theorem: (ER, Stong, Wang) Fusion graphs are:



Property F

A modular category \mathbf{D} has property F
if the subgroup:

$$\Phi(\mathbf{B}_n) \subset \mathbf{GL}(\text{End}(\mathbf{V}^{\otimes n}))$$

is *finite* for *all* objects \mathbf{V} in \mathbf{D} .

Property F Conjecture

Conjecture: (ER)

D a UMTC has

property $F \Leftrightarrow \dim(X_i)^2 \in \mathbb{Z}$

for all simple $X_i \in \mathbf{D}$

Property F Hierarchy



$\dim(X_i)^2 \in \mathbf{Z}$	<ul style="list-style-type: none">• $\mathbf{C}(\mathfrak{so}_{2k+1}, 4k+2)$• $\mathbf{D}(\mathbf{TY})$?
$\dim(X_i) \in \mathbf{Z}$	<ul style="list-style-type: none">• \mathbf{DG}• f.d. q-Hopf?
$\dim(X_i) = 1$	<ul style="list-style-type: none">• \mathbf{DA}, \mathbf{A} abelian• $\mathbf{C}(\mathfrak{sl}_n, q, n+1)$

Prop. F vs. Invariant Complexity

Construct.	Prop. F ?	Invariant	Complexity
$C(\mathfrak{sl}_2, q, L)$	$L=2,3,4,6$	Arf, $H_1(M, \mathbf{Z}/3\mathbf{Z})$ Jones	P if $L=2,3,4,6$ #P-hard else
$C(\mathfrak{sl}_n, q, L)$	$L=n+1, 4, 6$	classical HOMFLYPT	P if $L=n+1, 4, 6$ #P-hard else
$C(\mathfrak{so}_{2k+1}, q, L)$	$L=4k+2$	$H_1(M, \mathbf{Z}/N\mathbf{Z})$ Kauffman	P if $L=4k+2$ #P-hard else
Rep(D G)	Yes	$\#\{\text{Hom}(\pi_1(K'), G)\}$	P if G solvable ? else?

Exotic MTCs

- Observation: All MTCs $\text{rank} \leq 4$ have **quantum group** realizations.
- Conjecture (Moore & others): **All** MTCs “come from” **quantum groups**.
- Probably false: counterexamples called **EXOTIC**

Two Potentially Exotic MTCs

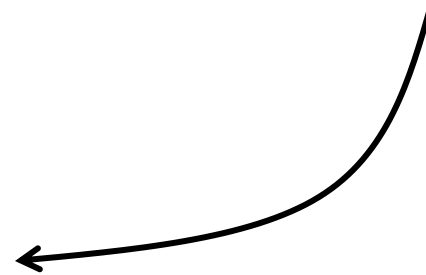
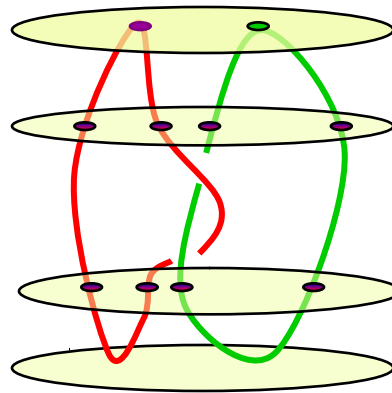
- $D(\mathfrak{E})$ a rank 10 category: “doubled $\frac{1}{2}E_6$ ”
- $D(\mathfrak{H})$ a rank 12 category: “doubled $\frac{1}{2}\text{Haagerup}$ ” related to subfactor of index $\frac{1}{2}(5 + \sqrt{13})$
- Analogous to finite simple **sporadic groups**
- Difficult to make precise & prove...

Physical Feasibility

Realizable TQC

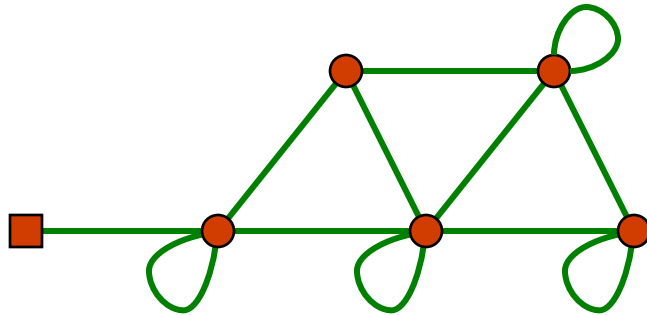


B_n action Unitary



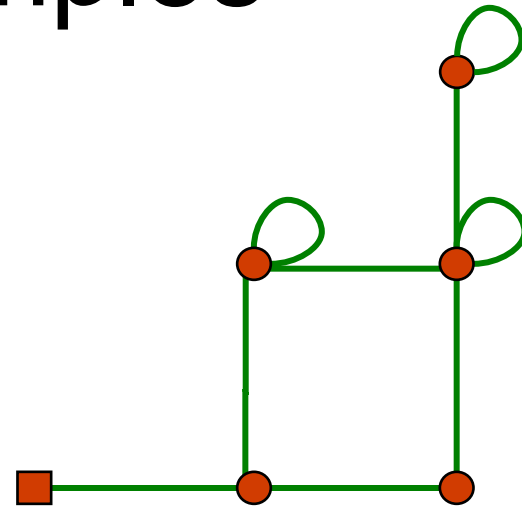
i.e. *Unitary* Modular Category

Two Examples



Unitary, for some q

Lie type G_2 $q^{21}=-1$



Never Unitary

Half of Lie type B_2 $q^9=-1$

Known for quantum group categories

Thanks!