0.1. **Orthogonally diagonalizing Symmetric Matrices.** If \( A = (a_{ij}) \) is a (not necessarily square) matrix, the transpose of \( A \) denoted \( A^T \) is the matrix with \((i, j)\) entry \((a_{ji})\). It is gotten from \(A\) by exchanging the \(i\)th row with the \(i\)th column, or by “reflecting across the diagonal.” Throughout this note, all matrices will have real entries.

The following are properties satisfied by the transpose.

**Lemma.**
1. \((AB)^T = B^T A^T\)
2. \((A^T)^{-1} = (A^{-1})^T\)
3. \((A + B)^T = A^T + B^T\)

**Definition.** A matrix \( A \) is called **symmetric** if \( A = A^T \).

Symmetric matrices have very nice properties. In particular they are **orthogonally diagonalizable**. This means that if \( A \) is symmetric, there is a basis \( \mathcal{B} = \{v_1, \ldots, v_n\} \) for \( \mathbb{R}^n \) consisting of eigenvectors for \( A \) so that the vectors in \( \mathcal{B} \) are pairwise orthogonal! Another way of saying this is that there exists a matrix \( P \) (with real entries) such that \( PP^T = P^T P = I \) and \( P^T AP \) is a diagonal matrix. A matrix \( P \) such that \( P^{-1} = P^T \) is called an **orthogonal** matrix.

Let \( x \cdot y \) denote the usual dot product on \( \mathbb{R}^n \). Notice this can be written \( x \cdot y = x^T y \), that is ordinary matrix multiplication of the “row vector” \( x^T \) and the column matrix \( y \).

In particular, if \( A \) is symmetric \((Ax) \cdot y = x \cdot (Ay)\).

**Fact.** Symmetric matrices always have real eigenvalues (and hence real eigenvectors).

Moreover,

**Theorem.** If \( A \) is symmetric, then eigenvectors of \( A \) with distinct eigenvalues are orthogonal.

**Proof** Let \( v \) and \( w \) be eigenvectors for a symmetric matrix \( A \) with different eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Then \( Av \cdot w = \lambda_1 (v \cdot w) \) but also \( Av \cdot w = (v \cdot Aw) = \lambda_2 (v \cdot w) \), so that \( \lambda_1 (v \cdot w) = \lambda_2 (v \cdot w) \), and since \( \lambda_1 \neq \lambda_2 \), we must have \((v \cdot w) = 0\).

To understand why a symmetric matrix is orthogonally diagonalizable we must use mathematical induction, so we won’t bother. However, we have an algorithm for finding an orthonormal basis of eigenvectors. Let \( A \) be an \( n \times n \) symmetric matrix.

1. If \( A \) has \( n \) distinct eigenvalues, then by the theorem above the corresponding eigenvectors are automatically orthogonal. To get orthonormality, just divide each eigenvector by its length.
2. Suppose \( A \) has a repeated eigenvalue \( \lambda \). Find a basis (of eigenvectors) \( \{v_1, v_2, \ldots, v_k\} \) for \( N(A - \lambda I) \). Since \( A \) is diagonalizable, there will the same number of eigenvectors corresponding to eigenvalue \( \lambda \) as the number of times \( \lambda \) appears as a root of the characteristic polynomial of \( A \). Apply the Gram-Schmidt process to get an orthogonal basis of eigenvectors \( \{x_1, x_2, \ldots, x_k\} \).
3. Repeat the above step for each repeated eigenvalue. Putting all of these bases for \( N(A - \lambda_i I) \) together we will have an orthonormal basis.
0.2. An Application. Consider the surface defined by

\[ G := \{(x, y, z) : Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + ax + by + cz = d\} \]

For example, if \( D = E = F = a = b = c = 0 \) and \( A = B = C = d = 1 \) this describes a sphere of radius 1 centered at the origin. \( x^2 + y^2 - z^2 = 1 \) describes a hyperboloid of one sheet, \( x^2 - y^2 - z^2 = 1 \) is a hyperboloid of 2 sheets, \( x^2 + y^2 = 1 \) is a cylinder, \( x^2 + y^2 = z \) is a paraboloid etc. In general if the “cross-terms” \( D, E \) and \( F \) are non-zero, it is difficult to determine the shape of the surface \( G \).

Define \( S := \begin{pmatrix} A & D/2 & E/2 \\ D/2 & B & F/2 \\ E/2 & F/2 & C \end{pmatrix} \), and set \( X := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \). Observe that the above equation can be written:

\[ X^T SX + \begin{pmatrix} a & b & c \end{pmatrix} X = d. \]

Since \( S \) is a symmetric matrix, there exists a \( P \) so so that \( P^T SP \) is diagonal, say with diagonal entries \( \alpha, \beta \) and \( \gamma \). Let \( Y := \begin{pmatrix} u \\ v \\ w \end{pmatrix} \), and make the change of variables \( X = PY \).

An important fact about orthogonal matrices: they “preserve length and angles.” This follows from the following exercise:

This implies that the effect of the change of variables \( X = PY \) on the surface \( G \) is just a rigid rotation. Let us work out the substitution:

\[ X^T SX + \begin{pmatrix} a & b & c \end{pmatrix} X = d \]

becomes

\[ Y^T P^T SPY + \begin{pmatrix} a & b & c \end{pmatrix} PY = d. \]

Since \( P^T SP \) is a diagonal matrix, the term \( Y^T P^T SPY = \alpha u^2 + \beta v^2 + \gamma w^2 \), that is, we have removed the \( xy, xz \) and \( yz \) terms!

To determine the shape of

\[ \alpha u^2 + \beta v^2 + \gamma w^2 + a'u + b'v + c'w = d \]

we just complete the square:

\[ \alpha(u^2 + \frac{a'}{\alpha} u + \left(\frac{a'}{2\alpha}\right)^2) + \beta v^2 + \gamma w^2 + b'v + c'w = d + \frac{(a')^2}{4\alpha} \]

which becomes:

\[ \alpha(u + \frac{a'}{2\alpha})^2 + \beta v^2 + \gamma w^2 + b'v + c'w = d + \frac{(a')^2}{4\alpha}. \]

Of course you must complete the squares in \( v \) and \( w \) as well. You will end up with an equation of the form \( L(u - u_0)^2 + M(v - v_0)^2 + N(w - w_0)^2 = Q \), the shape of which can be easily determined.

Notice that the shape of \( G \) is the same as the shape of the surface described by the final equation. The only difference is that \( G \) is “tilted” in space.
Exercise (1). Verify that \((A^T)^{-1} = (A^{-1})^T\), using the fact that \((AB)^T = B^T A^T\).

Exercise (2). Show that for a square matrix \(A\), \((Ax) \cdot y = x \cdot (A^T y)\).

Exercise (3). Assume that \(P\) is orthogonal. Show that \(v \cdot w = (Pv) \cdot (Pw)\).

Exercise (4). Determine the shape of the curve: \(x^2 - 2y^2 + 8xy = 4\) by using the above technique. Observe that the matrix \(S\) will be \(2 \times 2\).