

0.1. Orthogonally diagonalizing Symmetric Matrices. If $A = (a_{ij})$ is a (not necessarily square) matrix, the transpose of A denoted A^T is the matrix with (i, j) entry (a_{ji}) . It is gotten from A by exchanging the i th row with the i th column, or by “reflecting across the diagonal.” Throughout this note, all matrices will have real entries.

The following are properties satisfied by the transpose.

- Lemma.**
- (1) $(AB)^T = B^T A^T$
 - (2) $(A^T)^{-1} = (A^{-1})^T$
 - (3) $(A + B)^T = A^T + B^T$

Definition. A matrix A is called **symmetric** if $A = A^T$.

Symmetric matrices have very nice properties. In particular they are *orthogonally diagonalizable*. This means that if A is symmetric, there is a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ for \mathbb{R}^n consisting of eigenvectors for A so that the vectors in \mathcal{B} are pairwise orthogonal! Another way of saying this is that there exists a matrix P (with real entries) such that $PP^T = P^T P = I$ and $P^T A P$ is a diagonal matrix. A matrix P such that $P^{-1} = P^T$ is called an *orthogonal* matrix.

Let $x \cdot y$ denote the usual dot product on \mathbb{R}^n . Notice this can be written $x \cdot y = x^T y$, that is ordinary matrix multiplication of the “row vector” x^T and the column matrix y .

In particular, if A is symmetric $(Ax) \cdot y = x \cdot (Ay)$.

Fact. Symmetric matrices always have real eigenvalues (and hence real eigenvectors).

Moreover,

Theorem. *If A is symmetric, then eigenvectors of A with distinct eigenvalues are orthogonal.*

Proof Let v and w be eigenvectors for a symmetric matrix A with different eigenvalues λ_1 and λ_2 . Then $Av \cdot w = \lambda_1(v \cdot w)$ but also $Av \cdot w = (v \cdot Aw) = \lambda_2(v \cdot w)$, so that $\lambda_1(v \cdot w) = \lambda_2(v \cdot w)$, and since $\lambda_1 \neq \lambda_2$, we must have $(v \cdot w) = 0$.

To understand why a symmetric matrix is orthogonally diagonalizable we must use mathematical induction, so we won't bother. However, we have an algorithm for finding an orthonormal basis of eigenvectors. Let A be an $n \times n$ symmetric matrix.

- (1) If A has n distinct eigenvalues, then by the theorem above the corresponding eigenvectors are automatically orthogonal. To get orthonormality, just divide each eigenvector by its length.
- (2) Suppose A has a repeated eigenvalue λ . Find a basis (of eigenvectors) $\{v_1, v_2, \dots, v_k\}$ for $N(A - \lambda I)$. Since A is diagonalizable, there will be the same number of eigenvectors corresponding to eigenvalue λ as the number of times λ appears as a root of the characteristic polynomial of A . Apply the Gram-Schmidt process to get an orthogonal basis of eigenvectors $\{x_1, x_2, \dots, x_k\}$.
- (3) Repeat the above step for each repeated eigenvalue. Putting all of these bases for $N(A - \lambda_i I)$ together we will have an orthonormal basis.

0.2. **An Application.** Consider the surface defined by

$$G := \{(x, y, z) : Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + ax + by + cz = d\}$$

. For example, if $D = E = F = a = b = c = 0$ and $A = B = C = d = 1$ this describes a sphere of radius 1 centered at the origin. $x^2 + y^2 - z^2 = 1$ describes a hyperboloid of one sheet, $x^2 - y^2 - z^2 = 1$ is a hyperboloid of 2 sheets, $x^2 + y^2 = 1$ is a cylinder, $x^2 + y^2 = z$ is a paraboloid etc. In general if the “cross-terms” D, E and F are non-zero, it is difficult to determine the shape of the surface G .

Define $S := \begin{pmatrix} A & D/2 & E/2 \\ D/2 & B & F/2 \\ E/2 & F/2 & C \end{pmatrix}$, and set $X := \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Observe that the above equation can be written:

$$X^T S X + \begin{pmatrix} a & b & c \end{pmatrix} X = d.$$

Since S is a symmetric matrix, there exists a P so so that $P^T S P$ is diagonal, say with diagonal entries α, β and γ . Let $Y = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, and make the change of variables $X = P Y$.

An important fact about orthogonal matrices: they “preserve length and angles.” This follows from the following exercise:

This implies that the effect of the change of variables $X = P Y$ on the surface G is just a rigid rotation. Let us work out the substitution:

$$X^T S X + \begin{pmatrix} a & b & c \end{pmatrix} X = d$$

becomes

$$Y^T P^T S P Y + \begin{pmatrix} a & b & c \end{pmatrix} P Y = d.$$

Since $P^T S P$ is a diagonal matrix, the term $Y^T P^T S P Y = \alpha u^2 + \beta v^2 + \gamma w^2$, that is, we have removed the xy, xz and yz terms!

To determine the shape of

$$\alpha u^2 + \beta v^2 + \gamma w^2 + a'u + b'v + c'w = d$$

we just complete the square:

$$\alpha \left(u^2 + \frac{a'}{\alpha} u + \left(\frac{a'}{2\alpha} \right)^2 \right) + \beta v^2 + \gamma w^2 + b'v + c'w = d + \frac{(a')^2}{4\alpha}$$

which becomes:

$$\alpha \left(u + \frac{a'}{2\alpha} \right)^2 + \beta v^2 + \gamma w^2 + b'v + c'w = d + \frac{(a')^2}{4\alpha}.$$

Of course you must complete the squares in v and w as well. You will end up with an equation of the form $L(u - u_0)^2 + M(v - v_0)^2 + N(w - w_0)^2 = Q$, the shape of which can be easily determined.

Notice that the shape of G is the same as the shape of the surface described by the final equation. The only difference is that G is “tilted” in space.

Exercise (1). Verify that $(A^T)^{-1} = (A^{-1})^T$, using the fact that $(AB)^T = B^T A^T$.

Exercise (2). Show that for a square matrix A , $(Ax) \cdot y = x \cdot (A^T y)$.

Exercise (3). Assume that P is orthogonal. Show that $v \cdot w = (Pv) \cdot (Pw)$.

Exercise (4). Determine the shape of the curve: $x^2 - 2y^2 + 8xy = 4$ by using the above technique. Observe that the matrix S will be 2×2 .