0.1. **Orthogonally diagonalizing Symmetric Matrices.** If $A = (a_{ij})$ is a (not necessarily square) matrix, the transpose of A denoted A^T is the matrix with (i, j) entry (a_{ji}). It is gotten from A by exchanging the ith row with the ith column, or by "reflecting across the diagonal." Throughout this note, all matrices will have real entries.

The following are properties satisfied by the transpose.

Lemma. (1) $(AB)^T = B^T A^T$
(2) $(A^T)^{-1} = (A^{-1})^T$
(3) $(A + B)^T = A^T + B^T$

Definition. A matrix A is called **symmetric** if $A = A^T$.

Symmetric matrices have very nice properties. In particular they are *orthogonally diagonalizable*. This means that if A is symmetric, there is a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors for A so that the vectors in $\mathcal{B}$ are pairwise orthogonal! Another way of saying this is that there exists a matrix P (with real entries) such that $PP^T = P^T P = I$ and $P^T AP$ is a diagonal matrix.

Definition. A matrix P such that $P^{-1} = P^T$ is called an **orthogonal** matrix.

Let $x \cdot y$ denote the usual dot product on $\mathbb{R}^n$. Notice this can be written $x \cdot y = x^T y$, that is ordinary matrix multiplication of the "row vector" x^T and the column matrix y.

In particular, if A is symmetric $(Ax) \cdot y = x \cdot (Ay)$.

Fact. Symmetric matrices always have real eigenvalues (and hence real eigenvectors).

Moreover,

Theorem. *If A is symmetric, then eigenvectors of A with distinct eigenvalues are orthogonal.*

Proof Let v and w be eigenvectors for a symmetric matrix A with different eigenvalues λ_1 and λ_2. Then $Av \cdot w = \lambda_1(v \cdot w)$ but also $Av \cdot w = (v \cdot Aw) = \lambda_2(v \cdot w)$, so that $\lambda_1(v \cdot w) = \lambda_2(v \cdot w)$, and since $\lambda_1 \neq \lambda_2$, we must have $(v \cdot w) = 0$.

To understand why a symmetric matrix is orthogonally diagonalizable we must use mathematical induction, so we won't bother. However, we have an algorithm for finding an orthonormal basis of eigenvectors. Let A be an $n \times n$ symmetric matrix.

(1) If A has n distinct eigenvalues, then by the theorem above the corresponding eigenvectors are automatically orthogonal. To get orthonormality, just divide each eigenvector by its length.
(2) Suppose A has a repeated eigenvalue λ. Find a basis (of eigenvectors) $\{v_1, v_2, \ldots, v_k\}$ for $N(A - \lambda I)$. Since A is diagonalizable, there will the same number of eigenvectors corresponding to eigenvalue λ as the number of times λ appears as a root of the characteristic polynomial of A. Apply the Gram-Schmidt process to get an orthogonal basis of eigenvectors $\{x_1, x_2, \ldots, x_k\}$.
(3) Repeat the above step for each repeated eigenvalue. Putting all of these bases for $N(A - \lambda_i I)$ together we will have an orthonormal basis.

0.2. **Orthogonal Matrices.** Orthogonal matrices have useful properties as well. For example, if $v, w \in \mathbb{R}^n$ and we let θ be the angle between them, then $\cos(\theta) = v \cdot w/(||v|| \cdot ||w||)$. Exercise [3] below implies that if P is orthogonal then

$$Pv \cdot Pw/(||Pv|| \cdot ||Pw|| = \cos(\theta)$$

so that the linear transformation $f(x) = Px$ preserves length and preserves the cosine of the angle between any two vectors.

As we observed above, the P is an orthogonal matrix if and only if its columns form an orthonormal basis for $\mathbb{R}^n$. Let us figure out all real orthogonal 2×2 matrices. A matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is orthogonal if $PP^T = I$ so that

$$\begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$. The first condition implies that $(a, b) = (\cos(\theta), \sin(\theta))$ and $(c, d) = (\cos(\phi), \sin(\phi))$ for some angles $0 \leq \theta, \phi < 2\pi$. This is because the points (a, b) and (c, d) are on a circle of radius 1. Now the second equation:

$$0 = ac + bd = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) =$$
$$\frac{\cos(\theta - \phi) + \cos(\theta + \phi)}{2} + \frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{2} = \cos(\theta - \phi)$$

The equation $0 = \cos(x)$ implies that $x = \pi/2$ or $3\pi/2$, assuming $0 \leq x < 2\pi$, so that $\theta = \phi \pm \pi/2$, so $(c, d) = (\cos(\phi), \sin(\phi)) = \pm(-\sin(\theta), \cos(\theta))$. Putting it all together we have two types of orthogonal 2×2 matrices:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

Notice that $\det(P) = \pm 1$, which is a general fact you will prove in the exercises. Since we may choose θ to be any angle in $[0, 2\pi)$ so there are infinitely many 2×2 orthogonal matrices. The linear transformation $f(x) = Px$ rotates the vector x through an angle of θ or $-\theta$ depending on $\det(P)$. Another way of seeing that the above are the only possible 2×2 orthogonal matrices is to observe that, in $\mathbb{R}^2$, for any fixed vector x with $||x|| = 1$ there are exactly two vectors y with $||y|| = 1$ and $x \cdot y = 0$.

Challenge problem: Define $Q(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$ and $P(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. (Notice that $\det(P(\theta)) = 1$ while $\det(Q(\theta)) = -1$.) Show that $P(\theta)P(\phi) = P(\theta + \phi)$, $Q(\theta)Q(\phi) = P(\phi - \theta)$ and $P(\theta)Q(\phi) = Q(\phi - \theta)$.

The $n \times n$ orthogonal matrices of determinant $+1$ can also be visualized as "rigid motions" in space. In $\mathbb{R}^3$ the effect of muliplying a vector x by such an orthogonal matrix is to rotate x through two angles in succession. Notice that the matrix that switches the x-axis and the y-axis while fixing the z-axis is *not* a rigid motion, but has determinant -1.

Exercise (1). Verify that $(A^T)^{-1} = (A^{-1})^T$, using the fact that $(AB)^T = B^T A^T$.

Exercise (2). Show that for a square matrix A, $(Ax) \cdot y = x \cdot (A^T y)$.

Exercise (3). Assume that P is orthogonal. Show that $v \cdot w = (Pv) \cdot (Pw)$.

Exercise (4). Suppose that $\{v_1, \ldots, v_k\}$ is an orthogonal set in $\mathbb{R}^n$. Show that the set $\{Pv_1, \ldots, Pv_k\}$ is also orthogonal if P is an orthogonal matrix.

Exercise (5). Assume that P is orthogonal and symmetric. Show that $P^2 = I$.

Exercise (6). Show that if P and Q are orthogonal matrices, then so is PQ. Show that $\det(P) = \pm 1$.

Exercise (7). Show that the set, S of symmetric matrices is a subspace of $\mathcal{M}_{n,n}$. Determine $\dim(S)$.

Exercise (8). Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find an *orthogonal* matrix P so that PBP^{-1} is diagonal.