

Exercise (1). Verify that $(A^T)^{-1} = (A^{-1})^T$, using the fact that $(AB)^T = B^T A^T$.

Solution. We must show that $(A^T)(A^{-1})^T = I$. Using the fact above we have: $(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$.

Exercise (2). Show that for a square matrix A , $(Ax) \cdot y = x \cdot (A^T y)$.

Solution. Converting to matrix multiplication and using associativity, we have: $(Ax) \cdot y = (Ax)^T y = (x^T A^T) y = x^T (A^T y) = x \cdot (A^T y)$.

Exercise (3). Assume that P is orthogonal. Show that $v \cdot w = (Pv) \cdot (Pw)$.

Solution. By exercise (2) $(Pv) \cdot (Pw) = v \cdot (P^T Pw)$, and by orthogonality of P we have $P^T P = I$ so that $v \cdot (P^T Pw) = v \cdot Iw = v \cdot w$.

Exercise (4). Suppose that $\{v_1, \dots, v_k\}$ is an orthogonal set in \mathbb{R}^n . Show that the set $\{Pv_1, \dots, Pv_k\}$ is also orthogonal if P is an orthogonal matrix.

Solution. We must show that $Pv_i \cdot Pv_j = 0$ if $i \neq j$. From Exercise (3) we have $Pv_i \cdot Pv_j = v_i \cdot v_j$ since P is orthogonal. Since $\{v_1, \dots, v_k\}$ is an orthogonal set, $v_i \cdot v_j = 0$ if $i \neq j$.

Exercise (5). Assume that P is orthogonal and symmetric. Show that $P^2 = I$.

Solution. Since P is symmetric, $P = P^T$, so that P orthogonal means $P^2 = PP^T = I$.

Exercise (6). Show that if P and Q are orthogonal matrices, then so is PQ . Show that $\det(P) = \pm 1$.

Solution. We must show that $(PQ)(PQ)^T = I$. Since P and Q are orthogonal we compute: $(PQ)(PQ)^T = (PQ)Q^T P^T = PIP^T = PP^T = I$. Recalling that $\det(A^T) = \det(A)$, from the equation $PP^T = I$ we obtain $\det(I) = 1 = \det(PP^T) = \det(P) \det(P^T) = \det(P)^2$, so that $\det(P) = \pm 1$.

Exercise (7). Show that the set, S of symmetric matrices is a subspace of $\mathcal{M}_{n,n}$. Determine $\dim(S)$.

Solution. We must show that if A and B are symmetric then $A+B$ and rA are symmetric for any $r \in \mathbb{R}$. But this is clear, since $(A+B)^T = (A^T + B^T) = A+B$, and $(rA)^T = rA^T = rA$. To determine the dimension of S , verify that a basis for S is $\{E_{kk}, E_{ij} + E_{ji}\}$ where $1 \leq k \leq n$ and $1 \leq i < j \leq n$ (so we don't count $E_{12} + E_{21}$ twice). Then counting we get $n + (n-1) + \dots + 2 + 1$ basis elements, which adds up to $(n+1)(n)/2$. So, for example, when $n = 2$ $\dim(S) = 3$.

Exercise (8). Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find an orthogonal matrix P so that PBP^{-1} is diagonal.

Solution. We see that the eigenvalues of B are 0 (twice) and 3. A length 1 eigenvector for $\lambda = 3$ is $u_3 := \frac{1}{\sqrt{3}}(1, 1, 1)$. The eigenvalue $\lambda = 0$ corresponds to vectors in $\text{Null}(B)$,

which we find by solving $Bx = 0$. We get $(x, y, z) = (-t - s, t, s)$, so that $(-1, 1, 0)$ and $(-1, 0, 1)$ are eigenvectors with eigenvalue 0. We must orthonormalize: $u_1 = \frac{1}{\sqrt{2}}(-1, 1, 0)$,

$$x_2 = (-1, 0, 1) - u_1 \cdot (-1, 0, 1)u_1 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = \frac{1}{2}(-1, -1, 2)$$

so that $u_2 = x_2/\|x_2\| = \frac{1}{\sqrt{6}}(-1, -1, 2)$. Thus we have $P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$