1. SOLVING 2ND ORDER DIFFERENCE EQUATIONS

Difference equations are discrete versions of differential equations. While linear, homogeneous differential equations deal with the differential operator $D = \frac{d}{dx}$ on the vectorspace $C^\infty(\mathbb{R})$ of infinitely differentiable $\mathbb{R}$-valued functions, linear homogeneous *difference* equations deal with the "left-shift" linear operator L on $\mathbb{R}^\infty$. Despite these very different looking situations, solving difference and differential equations employ the same basic technique.

1.1. **Example: Fibonacci sequence.**

Consider the Fibonacci sequence

$$\mathbf{f} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots).$$

Each term f_i of this sequence is defined to be the sum of the previous two $f_i = f_{i-1} + f_{i-2}$, after defining the first two terms to be 0 and 1. To find the 100th member of this sequence from this definition would require computing the first 99. This would take a long time. So instead we would like a formula for the nth term. Why would we want to do this? One reason would be to compute the limit of ratios $\lim_{i\to\infty} f_{i+1}/f_i$ to see the rate of growth of the sequence. How do we do this? First we notice that f_i satisfies the *difference equation* $a_{n+1} - a_n - a_{n-1} = 0$. In linear algebraic terms this means that $\mathbf{f}$ is in the nullspace of $L^2 - L - I$ where L is the "left-shift" linear operator on $\mathbb{R}^\infty$:

$$L(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots).$$

So, if we can find a nice basis for $N(L^2 - L - I)$, we can express $\mathbf{f}$ in this basis to hopefully obtain a formula. Factoring the equation $x^2 - x - 1 = (x - \tau_1)(x - \tau_2)$ where $\tau_1 = \frac{1+\sqrt{5}}{2} = \tau$ and $\tau_2 = \frac{1-\sqrt{5}}{2} = 1 - \tau$, we see that if any sequence $\mathbf{a}$ satisfies $(L - \tau_1 I)\mathbf{a} = 0$ then also

$$(L^2 - L - I)\mathbf{a} = (L - \tau_2 I)(L - \tau_1 I)\mathbf{a} = (L - \tau_2 I)\mathbf{0} = \mathbf{0},$$

and similarly for sequences $\mathbf{b}$ satisfying $(L - \tau_1 I)\mathbf{b} = 0$. To find such sequences is easy, since $(L - \tau_1 I)\mathbf{a} = 0$ is the same as $L\mathbf{a} = \tau_1\mathbf{a}$. This equation has infinitely many solutions all of which are of the form $a_0(1, \tau_1, (\tau_1)^2, \ldots)$. Similarly, $\mathbf{b} = b_0(1, \tau_2, (\tau_2)^2, \ldots)$. That is, $\mathbf{T_1} := (1, \tau_1, (\tau_1)^2, (\tau_1)^3, \ldots)$ is an eigenvector for L with eigenvalue τ_1 with a similar statement for $\mathbf{T_2} := (1, \tau_2, (\tau_2)^2, \ldots)$. Since these two sequences are linearly independent and any sequence in the nullspace of $L^2 - L - I$ is determined by its first two terms it can be seen that $\{T_1, T_2\}$ is a basis for $N(L^2 - L - I)$. Thus there are constants a_0 and b_0 so that $\mathbf{f} = a_0 T_1 + b_0 T_2$. Looking at the first two terms of this equation we get: $0 = a_0 + b_0$ and $1 = a_0\tau + b_0(1 - \tau)$. Plugging in $\tau = \frac{1+\sqrt{5}}{2}$ and solving we get $a_0 = -b_0 = 1/\sqrt{5}$, so that:

$$\mathbf{f} = (0, 1, 1, \ldots, f_i, \ldots) = \frac{1}{\sqrt{5}}(1, \tau, \tau^2, \ldots, \tau^i, \ldots) - \frac{1}{\sqrt{5}}(1, (1-\tau), (1-\tau)^2, \ldots, (1-\tau)^i, \ldots)$$

so that

$$f_i = \frac{\tau^i - (1-\tau)^i}{\sqrt{5}}.$$

Exercise (1). Compute the limit: $\lim_{i\to\infty} \frac{f_{i+1}}{f_i}$ if it exists.

1.2. **Recollection of Differential Equations.** Consider the 2nd order differential equation

$$f'' + bf' + f = 0 \tag{1}$$

Recall the technique for solving such an equation: factor the polynomial x^2+bx+c to get solutions $\lambda_1, \lambda_2 = \frac{-b\pm\sqrt{b^2-4c}}{2}$. If $\lambda_1 \neq \lambda_2$ then the general solution is $f(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, as $De^{\lambda t} = \lambda e^{\lambda t}$ where the scalars c_1 and c_2 are determined by initial conditions. For example, if $f(0) = a$ and $f'(0) = b$ then one solves $c_1 + c_2 = a$ and $\lambda_1 c_1 + \lambda_2 c_2 = b$. Since $\lambda_1 \neq \lambda_2$ one is guaranteed *unique* solutions for c_1 and c_2, as the matrix $\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$ has determinant $\lambda_1 - \lambda_2$.

If $\lambda_1 = \lambda_2 = \lambda$, then the general solution is $f(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$, where again c_1 and c_2 are determined by initial conditions $f(0) = a$ and $f'(0) = b$.

Exercise (2). Verify that $f(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$ satisfies the differential equation above in the case that $\lambda_1 = \lambda_2$. Then show that the initial conditions $f(0) = a$ and $f'(0) = b$ uniquely determine c_1 and c_2.

Example. The differential equation $f'' - 6f' + 9f = 0$ with initial conditions $f(0) = 1$, $f'(0) = 2$ has solution $(1-t)e^{3t}$.

1.3. **Difference Equations.** Now consider the difference equation analogous to (1):

$$a_n + ba_{n-1} + ca_{n-2} = 0 \tag{2}$$

which is equivalent to $a_n = -ba_{n-1} - c_{n-2}$. Suppose we are given the first two terms a_0 and a_1 of a sequence $\mathbf{a}$ satisfying the difference equation (2). The solution follows the same general pattern of the differential equations above. Let λ_1, λ_2 be the two solutions to $x^2 + bx + c = 0$. First let us suppose that $\lambda_1 \neq \lambda_2$. Define $\lambda_1 = (1, \lambda_1, (\lambda_1)^2, \ldots)$ and $\lambda_2 = (1, \lambda_2, (\lambda_2)^2, \ldots)$. Then the general solution to (2) is $\mathbf{a} = c_1\lambda_1 + c_2\lambda_2$ where c_1 and c_2 can be uniquely determined from the first two terms of the sequence a_0 and a_1.

If $\lambda_1 = \lambda_2 = \lambda$, then the solution is similar to the repeated root case for the differential equation. The solution is of the form:

$$(a_0, a_1, a_2, \cdots, a_n, \cdots) = c_1(1, \lambda, \lambda^2, \cdots, \lambda^n \cdots) + c_2(0, \lambda, 2\lambda^2, \cdots, n\lambda^n, \cdots).$$

Exercise (3). Suppose that $4a_n = 4a_{n-1} - a_{n-2}$, and $a_0 = a_1 = 1$. Find a formula for the nth term of the sequence. What is $\lim_{n\to\infty} a_n$?

Exercise (4). Find a difference equation satisfied by the sequence $(0, 1, 2, 3, \ldots)$.