

[This file contains the computations relevant to the paper "On the classification of non-self-dual modular categories" by Hong and Rowell.

[> with(LinearAlgebra): with(Groebner):

[Using the Galois argument assuming that (012)(34) is the generator of the group, one reduces (by excluding some possibilities or by permuting the labels) to the following S-matrix. We assume d,f,g are >1.

> S:=Matrix(5,5,[[1,d,f,g,g],[d,-f,-1,g,g],[f,-1,d,-g,-g],[g,g,-g,h1+I*h2,h1-I*h2],[g,g,-g,h1-I*h2,h1+I*h2]]);

$$S := \begin{bmatrix} 1 & d & f & g & g \\ d & -f & -1 & g & g \\ f & -1 & d & -g & -g \\ g & g & -g & h1+h2I & h1-h2I \\ g & g & -g & h1-h2I & h1+h2I \end{bmatrix}$$

> C:=Matrix(5,5,[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,0,1],[0,0,0,1,0]]);

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

[These relations describe the condition that S^2 is proportional to the "charge conjugation matrix."

> Srels:=factor(convert(evalm(S^2-K^2*C),set));

Srels := {(1-f+2h1+d)g, -(1-f+2h1+d)g, 3g^2+2h1^2-2h2^2, d-df-f+2g^2, f-d+df-2g^2, 3g^2+2h1^2+2h2^2-K^2, 1+d^2+f^2+2g^2-K^2}

> factor(Basis('union'(Srels, {1+d-f+2*h1}), lexdeg([K,d,f,g],[h2,h1])));

[1-f+2h1+d, 3g^2+2h1^2-2h2^2, -3f+3f^2-6fhl+3+6h1+4h1^2-4h2^2, (K-2h2)(K+2h2)]

[We may assume h2>0 by conjugating if necessary.

> factor(Basis('union'(Srels, {1+d-f+2*h1, K-2*h2}), tdeg(K,h2,h1,d,f,g))));

[1-f+2h1+d, K-2h2, f-d+df-2g^2, 4h2^2-1-d^2-f^2-2g^2]

[These relations come down to a single relation among the d,f and g (the other allow elimination of K, h1 and h2).

> orthrel:=-d+d*f+f-2*g^2;

orthrel := f-d+df-2g^2

[We use the various symmetries of the N_{i,j}^k to write down the fusion matrices in terms of just 14 variables. Note that M4 is just the transpose of M3.

> M1:=Matrix([[0,1,0,0,0],[1,n1,n2,n3,n3],[0,n2,n4,n5,n5],[0,n3,n5,n

6,n7],[0,n3,n5,n7,n6]]);

$$M1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & n1 & n2 & n3 & n3 \\ 0 & n2 & n4 & n5 & n5 \\ 0 & n3 & n5 & n6 & n7 \\ 0 & n3 & n5 & n7 & n6 \end{bmatrix}$$

> M2:=Matrix([[0,0,1,0,0],[0,n2,n4,n5,n5],[1,n4,n8,n9,n9],[0,n5,n9,n10,n11],[0,n5,n9,n11,n10]]);

$$M2 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & n2 & n4 & n5 & n5 \\ 1 & n4 & n8 & n9 & n9 \\ 0 & n5 & n9 & n10 & n11 \\ 0 & n5 & n9 & n11 & n10 \end{bmatrix}$$

> M3:=Matrix([[0,0,0,0,1],[0,n3,n5,n7,n6],[0,n5,n9,n11,n10],[1,n6,n10,n12,n13],[0,n7,n11,n14,n12]]);

$$M3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & n3 & n5 & n7 & n6 \\ 0 & n5 & n9 & n11 & n10 \\ 1 & n6 & n10 & n12 & n13 \\ 0 & n7 & n11 & n14 & n12 \end{bmatrix}$$

The matrices must commute, giving a set of diophantine equations.

> comrels:=`minus`(`union`(convert(evalm(M1*M2-M2*M1),set),convert(evalm(M1*M3-M3*M1),set),convert(evalm(M2*M3-M3*M2),set),convert(evalm(M3*Transpose(M3)-Transpose(M3)*M3),set)),{0}));

comrels := {n12 - n13, n13 - n12, n3 n6 + n5 n10 + n7 n13 - n3 n7 - n5 n11 - n7 n14, n3 n7 + n5 n11 + n7 n14 - n3 n6 - n5 n10 - n7 n13, n6 n5 + n9 n10 + n11 n13 - n7 n5 - n9 n11 - n11 n14, n7 n5 + n9 n11 + n11 n14 - n6 n5 - n9 n10 - n11 n13, 1 + n6² + n10² + n13² - n7² - n11² - n14², 1 + n1 n4 + n2 n8 + 2 n3 n9 - n2² - n4² - 2 n5², n2² + n4² + 2 n5² - 1 - n1 n4 - n2 n8 - 2 n3 n9, n3² + n5² + 2 n7 n6 - n1 n7 - n2 n11 - n3 n12 - n3 n14, n5² + n9² + 2 n11 n10 - n4 n7 - n8 n11 - n9 n12 - n9 n14, n7² + n11² + n14² - 1 - n6² - n10² - n13², n1 n7 + n2 n11 + n3 n12 + n3 n14 - n3² - n5² - 2 n7 n6,

```

n4 n7 + n8 n11 + n9 n12 + n9 n14 - n52 - n92 - 2 n11 n10,
n1 n5 + n2 n9 + n3 n10 + n3 n11 - n3 n2 - n5 n4 - n6 n5 - n7 n5,
n2 n5 + n4 n9 + n5 n10 + n5 n11 - n3 n4 - n5 n8 - n6 n9 - n7 n9,
n2 n6 + n4 n10 + n5 n13 + n5 n12 - n3 n5 - n5 n9 - n6 n10 - n7 n11,
n2 n7 + n4 n11 + n5 n12 + n5 n14 - n3 n5 - n5 n9 - n6 n11 - n7 n10,
n3 n2 + n5 n4 + n6 n5 + n7 n5 - n1 n5 - n2 n9 - n3 n10 - n3 n11,
n3 n4 + n5 n8 + n6 n9 + n7 n9 - n2 n5 - n4 n9 - n5 n10 - n5 n11,
n3 n5 + n5 n9 + n6 n10 + n7 n11 - n2 n6 - n4 n10 - n5 n13 - n5 n12,
n3 n5 + n5 n9 + n6 n11 + n7 n10 - n2 n7 - n4 n11 - n5 n12 - n5 n14,
n3 n6 + n5 n10 + n6 n13 + n7 n12 - n3 n7 - n5 n11 - n6 n12 - n7 n14,
n3 n7 + n5 n11 + n6 n12 + n7 n14 - n3 n6 - n5 n10 - n6 n13 - n7 n12,
n6 n5 + n9 n10 + n10 n13 + n11 n12 - n7 n5 - n9 n11 - n10 n12 - n11 n14,
n7 n5 + n9 n11 + n10 n12 + n11 n14 - n6 n5 - n9 n10 - n10 n13 - n11 n12,
1 + n1 n6 + n2 n10 + n3 n13 + n3 n12 - n32 - n52 - n62 - n72,
1 + n4 n6 + n8 n10 + n9 n13 + n9 n12 - n52 - n92 - n102 - n112,
n32 + n52 + n62 + n72 - 1 - n1 n6 - n2 n10 - n3 n13 - n3 n12,
n52 + n92 + n102 + n112 - 1 - n4 n6 - n8 n10 - n9 n13 - n9 n12 }

```

```
> indets(comrels);
```

```
{n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9}
```

The commutation relations alone leave 6 degrees of freedom.

```
> HilbertDimension(comrels, tdeg(n1, n10, n11, n12, n13, n14, n2, n3,
n4, n5, n6, n7, n8, n9));
```

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The Characteristic polynomials of M1, M2 and M3 are computed in two ways. First from the knowledge of the eigenvalues.

```
> ch1 := collect(expand(product(' (X-S[2,i]/S[1,i])', i=1..5)), X);
ch2 := collect(expand(product(' (X-S[3,i]/S[1,i])', i=1..5)), X);
ch3 := collect(expand(product(' (X-S[4,i]/S[1,i])', i=1..5)), X);
```

$$\begin{aligned}
ch1 := & -1 + X^5 + \left(-d + \frac{1}{f} - 2 + \frac{f}{d}\right)X^4 + \left(2d + 1 - \frac{2f}{d} - f - \frac{2}{f} + \frac{1}{d} - \frac{d}{f}\right)X^3 \\
& + \left(\frac{f}{d} - d + 2f + \frac{1}{f} + \frac{2d}{f} - 1 - \frac{2}{d}\right)X^2 + \left(-\frac{d}{f} + 2 - f + \frac{1}{d}\right)X
\end{aligned}$$

$$\begin{aligned}
ch2 := & 1 + X^5 + \left(-\frac{d}{f} + 2 - f + \frac{1}{d}\right)X^4 + \left(-2f + 1 + \frac{2}{d} - \frac{f}{d} - \frac{2d}{f} - \frac{1}{f} + d\right)X^3 \\
& + \left(2d + 1 - \frac{2f}{d} - f - \frac{2}{f} + \frac{1}{d} - \frac{d}{f}\right)X^2 + \left(d + 2 - \frac{f}{d} - \frac{1}{f}\right)X
\end{aligned}$$

$$ch3 := X^5 + \left(-\frac{2hl}{g} + \frac{g}{f} - \frac{g}{d} - g\right)X^4 + \left(-\frac{2hl}{f} + \frac{g^2}{d} + \frac{2hl}{d} + \frac{h2^2}{g^2} - \frac{g^2}{df} + \frac{hl^2}{g^2} - \frac{g^2}{f} + 2hl\right)X^3$$

$$+ \left(\frac{h^2}{fg} + \frac{g^3}{df} + \frac{2gh}{df} + \frac{hl^2}{fg} - \frac{h^2}{dg} + \frac{2gh}{f} - \frac{hl^2}{dg} - \frac{2gh}{d} - \frac{h^2}{g} - \frac{hl^2}{g} \right) X^2$$

$$+ \left(\frac{hl^2}{d} - \frac{hl^2}{df} - \frac{2g^2hl}{df} + \frac{h^2}{d} - \frac{hl^2}{f} - \frac{h^2}{df} - \frac{h^2}{f} \right) X + \frac{gh^2}{df} + \frac{ghl^2}{df}$$

The form of these polynomials imply further relations.

```
> {coeff(ch1,X,0)+1,coeff(ch2,X,0)-1,coeff(ch1,X,4)+coeff(ch2,X,1),coeff(ch1,X,1)-coeff(ch2,X,4),coeff(ch1,X,3)-coeff(ch2,X,2),coeff(ch1,X,2)+coeff(ch2,X,3)};
```

{0}

```
> factor(ch1); factor(ch2); factor(ch3);
```

$$\frac{(X-1)^2(fX+1)(X-d)(dX+f)}{df}$$

df

$$\frac{(X+1)^2(X-f)(dX+1)(fX-d)}{df}$$

df

$$\frac{(X-g)(g^2X^2-2ghlX+hl^2+h^2)(fX+g)(dX-g)}{g^2fd}$$

g^2fd

```
> p1:=CharacteristicPolynomial(M1,X);
```

```
p2:=CharacteristicPolynomial(M2,X);
```

```
p3:=CharacteristicPolynomial(M3,X);
```

```
> factor(p1); factor(p2);
```

$$(n7+X-n6)(X^2n1n4+2n5^2-n4n7-n4n6-X^3n6-X^3n4+X^2n6n4-2X^2n5^2+X^4-X^2+Xn7+Xn6-4Xn2n3n5-Xn6n1n4-Xn1n4n7-X^3n1-2X^2n3^2-X^2n2^2+Xn4-X^3n7+X^2n1n6+X^2n1n7+X^2n4n7+2Xn1n5^2+2Xn3^2n4+Xn6n2^2+Xn2^2n7)$$

$$(X-n10+n11)(X^2n2n8+2n5^2-n2n11-n2n10-2X^2n5^2+X^2n2n10+X^4-X^2-X^3n8-X^3n10+Xn10-2X^2n9^2+Xn11-Xn10n2n8-4Xn4n5n9-Xn2n8n11-X^3n2-X^2n4^2+Xn2-X^3n11+X^2n10n8+X^2n8n11+X^2n2n11+2Xn5^2n8+2Xn2n9^2+Xn10n4^2+Xn4^2n11)$$

Observe that the only linear term of p1 and p2 are (X-1) and (X+1) respectively.

```
> solve(-n10+n11+X=(X+1));
```

$$\{X=X, n10=n11-1, n11=n11\}$$

```
> solve(n7+X-n6=(X-1));
```

$$\{X=X, n6=n7+1, n7=n7\}$$

```
> linrels:={n11-(n10+1),n6-(n7+1)};
```

$$\text{linrels} := \{-1-n7+n6, -n10+n11-1\}$$

The relations implied by the coefficients of the Characteristic polynomials.

```
> chrels:={coeff(p1,X,0)+1,coeff(p2,X,0)-1,coeff(p1,X,4)+coeff(p2,X,1),coeff(p1,X,1)-coeff(p2,X,4),coeff(p1,X,3)-coeff(p2,X,2),coeff(p1,X,2)+coeff(p2,X,3)};
```

$$\text{chrels} := \{2n11n5^2-2n10n5^2-n11^2n2+n2n10^2-1,$$

```

n4 n6^2 - n4 n7^2 - 2 n5^2 n6 + 2 n5^2 n7 + 1, -2 n6 - n1 - n4 - 4 n5 n4 n9 n11 + n10^2 n2 n8
- 2 n5^2 n8 n10 - 2 n9^2 n2 n10 - 2 n2 n10 + 2 n9^2 n11 n2 + 4 n10 n4 n5 n9 + 2 n11 n5^2 n8
- n11^2 n2 n8 + 2 n5^2 + n11^2 n4^2 - n10^2 + n11^2 - n10^2 n4^2, 4 n2 n3 n5 n6 - 2 n3^2 n4 n6
- n1 n4 n7^2 - 4 n2 n3 n5 n7 + n1 n4 n6^2 + 2 n1 n5^2 n7 + 2 n5^2 - 2 n1 n5^2 n6 + n7^2 + 2 n3^2 n7 n4
- n6^2 + n7^2 n2^2 - n6^2 n2^2 - 2 n4 n6 + n2 + 2 n10 + n8, -2 n3^2 + n1 n4 - n2^2 + 2 n1 n6 + 2 n4 n6
- n7^2 - 1 - 2 n5^2 + n6^2 - n11^2 n2 + 2 n10 n2 n8 - 2 n10 n5^2 + 4 n4 n5 n9 - n11^2 n8 + 2 n9^2 n11
- 2 n5^2 n8 + n2 n10^2 - 2 n10 - 2 n2 n9^2 - 2 n10 n4^2 - n2 - 2 n10 n9^2 + n8 n10^2 + 2 n11 n5^2,
2 n6 n2^2 - 2 n3^2 n7 - 2 n5^2 n7 + 2 n3^2 n4 - n1 n6^2 + n1 n7^2 + 2 n1 n5^2 - n4 n6^2 + 2 n5^2 n6
- 4 n2 n3 n5 - 2 n6 n1 n4 + 2 n6 + 2 n3^2 n6 + n4 + n4 n7^2 - 1 - n4^2 + 2 n2 n10 - 2 n9^2 + n2 n8
+ 2 n8 n10 - 2 n5^2 - n11^2 + n10^2}
> nops(chrels);

```

6

We take all of the relations together and process them. We rename three of the variables u,v,t; for later use.

```

> allNrels:=subs({n10=t,n12=v,n14=u},`union`(linrels,comrels,chrels)
):

```

```

> factor(Basis(allNrels,plex(n1, n11, n13, n2, n3, n4, n5, n6, n7,
n8, n9,t,u,v)));

```

```

[(75 + 8 v^2)(-v + u)(2 u^2 t - u^2 - 4 u v - 3 v^2 - 2 t v^2 + 2 t + 1 + 4 t^2),
(5 u - v)(-v + u)(2 u^2 t - u^2 - 4 u v - 3 v^2 - 2 t v^2 + 2 t + 1 + 4 t^2),
(-v + u)(2 t + 14 - u^2 + v^2)(2 u^2 t - u^2 - 4 u v - 3 v^2 - 2 t v^2 + 2 t + 1 + 4 t^2),
(2 u^2 t - u^2 - 4 u v - 3 v^2 - 2 t v^2 + 2 t + 1 + 4 t^2)
(20 t^2 + 10 t - 10 u^2 t + 10 t v^2 + 5 - 104 u v + 99 v^2 + 5 u^2 + 5 u^4 - 10 u^2 v^2 + 5 v^4), -5 + 5 u^2
- 3 v^4 - 10 t - 24 v^2 + 18 v u t - v^3 u - 8 t v^2 + u^3 v - 10 u^2 t + 19 u v + 20 n9 v - 20 t^2 + 3 u^2 v^2
- 2 v^4 t + 2 v^3 u t - 2 u^3 v t - 4 u v t^2 + 4 v^2 t^2 + 2 u^2 t v^2, 15 u^2 + 3 v^4 - 30 t + 44 v^2 - 58 v u t
+ v^3 u + 28 t v^2 - u^3 v + 30 u^2 t - 59 u v + 60 n9 u - 60 t^2 - 3 u^2 v^2 - 15 + 2 v^4 t - 2 v^3 u t + 2 u^3 v t
+ 4 u v t^2 - 4 v^2 t^2 - 2 u^2 t v^2, -103 v^3 + 37 u^3 + 4 n9 + 27 v - 35 u + 54 v t - 70 u t - 37 v^2 u
+ 103 u^2 v + 62 v^2 u t + 108 v t^2 + 12 v^3 u^2 - 6 u^4 v - 2 u^5 + 16 n9 t^2 + 78 v u^2 t - 2 u v^4 + 4 u^3 v^2
+ 8 n9 t - 8 u^3 t v^2 - 4 u^4 t v + 4 u^5 t - 4 v^5 t - 78 v^3 t + 8 u^2 t v^3 + 4 v^4 u t - 62 u^3 t - 140 u t^2
- 6 v^5, -6 - 31 v^2 + 22 u v + 9 u^2 - 12 t + 3 v^4 + 10 v^3 u - 38 t v^2 - 10 u^3 v + 44 v u t - 3 u^4 - 6 u^2 t
- 24 t^2 + 2 v^4 t + 4 v^3 u t - 8 u^2 t v^2 - 16 v^2 t^2 - 4 u^3 v t + 16 u v t^2 + 6 u^4 t + 12 n9^2, 5 + 10 t
- 68 v^2 + 28 u v - 151 t v^2 + 66 v u t + 5 u^2 t + 34 v^4 + 118 v^3 u - 114 u^2 v^2 - 38 u^3 v - 152 v^2 t^2
+ 152 u v t^2 + 36 v^4 t + 4 v^3 u t - 116 u^2 t v^2 + 76 u^3 v t - 5 n8 + 40 v^2 n8, -6 v + 2 u - 12 v t + 4 u t
+ 3 v^3 + 7 v^2 u - 7 u^2 v - 3 u^3 - 12 v t^2 + 12 u t^2 + 3 v^3 t - v^2 u t - 7 v u^2 t + 5 u^3 t + 3 v n8 + n8 u,
-10 - 23 v^2 + 18 u v + 5 u^2 - 40 t + 9 v^4 + 18 v^3 u - 4 u^2 v^2 - 56 t v^2 - 18 u^3 v + 56 v u t - 5 u^4

```

$-40t^2 + 6v^4t + 4v^3ut - 16u^2tv^2 - 32v^2t^2 - 4u^3vt + 32uv^2t^2 + 10u^4t + 10n8 + 20n8t,$
 $-42v^3 + 8u^3 - 4n9 + 9v - 9u + 18vt - 18ut + 4v^2u + 30u^2v + 44v^2ut + 4n8n9 + 36vt^2$
 $+ 2v^3u^2 + u^4v + u^5 + 8vu^2t + 5uv^4 - 6u^3v^2 - 8n9t - 4u^3tv^2 + 6u^4tv - 2u^5t - 2v^5t$
 $- 32v^3t - 4u^2tv^3 + 6v^4ut - 20u^3t - 36ut^2 - 3v^5, 1 + v^2 - u^2 + 2n7 - 2t,$
 $-1 - 2t + v^2 - u^2 + 2n6, vt - ut + n5 - n9 + v - u, 29v^2 - 29uv + 13tv^2 - 28vut + 15u^2t$
 $+ 3v^4 + v^3u - 3u^2v^2 - u^3v - 4v^2t^2 + 4uv^2t + 2v^4t - 2v^3ut - 2u^2tv^2 + 2u^3vt - 15n8$
 $+ 15n4, v - u - 2n9 + 4vt - 4ut - v^3 + v^2u + u^2v - u^3 + 2n3, 15 + 43v^2 - 28uv - 15u^2$
 $+ 11tv^2 - 26vut + 15u^2t + 6v^4 + 2v^3u - 6u^2v^2 - 2u^3v - 8v^2t^2 + 8uv^2t + 4v^4t - 4v^3ut$
 $- 4u^2tv^2 + 4u^3vt - 15n8 + 15n2, n13 - v, -t + n11 - 1, 10 + 43v^2 - 28uv - 15u^2 - 4tv^2$
 $+ 4vut + 11v^4 - 8v^3u - 6u^2v^2 + 8u^3v - 5u^4 - 8v^2t^2 + 8uv^2t + 4v^4t - 4v^3ut - 4u^2tv^2$
 $+ 4u^3vt - 10n8 + 10n1]$

The first relation implies that either `diol` or `bad1` is zero. We will see that `bad1` cannot be 0.

```

> diol := 4*t^2 + 2*u^2*t + 2*t - 2*t*v^2 - u^2 - 4*u*v + 1 - 3*v^2;
> bad1 := u - v;

```

$$diol := 2u^2t - u^2 - 4uv - 3v^2 - 2tv^2 + 2t + 1 + 4t^2$$

$$bad1 := -v + u$$

```

> factor(Basis('union'(allNrels, {bad1}), plex(n1, n11, n13, n2, n3,
n4, n5, n6, n7, n8, n9, t, u, v)));

```

```

[-v + u, (2t + 1 + 4t^2)(4t^2 + 2t + 1 - 8v^2), -1 + 4n9v - 2t - 4t^2, (2t + 1 + 4t^2)(n9 - 2v),
-1 + 2n9^2 - 4t^2 - 2t, n8 - 1 - 2t, 2n7 + 1 - 2t, 2n6 - 1 - 2t, -n9 + n5, n4 - 1 - 2t, n3 - n9,
n2 - 2t, n13 - v, -t + n11 - 1, n1 - 2t]

```

```

> bad2 := 1 - 8*v^2 + 2*t + 4*t^2;

```

$$bad2 := 4t^2 + 2t + 1 - 8v^2$$

```

> 'mod'(bad2, 4);

```

$$2t + 1$$

```

> factor(Basis('union'(allNrels, {diol}), plex(n1, n11, n13, n2, n3,
n4, n5, n6, n7, n8, n9, t, u, v)));

```

```

[2u^2t - u^2 - 4uv - 3v^2 - 2tv^2 + 2t + 1 + 4t^2, v(-2v - vt + n9 + ut), u(-2v - vt + n9 + ut), u^2
- 3v^4 - u^4 - 15v^2 + 12vut + 2v^3u - 14tv^2 - 2u^3v + 2u^2t - 2uv + 4n9^2 + 2u^4t + 4u^2v^2
- 2v^4t + 4v^3ut - 4u^3vt, (8v^2 - 1)(-2v^2 - tv^2 + 2uv + 2vut + n8 - 2t - 1 - u^2t),
(3v + u)(-2v^2 - tv^2 + 2uv + 2vut + n8 - 2t - 1 - u^2t), -9v^2 - 6uv - u^2 - 4t - 3v^4 + 2v^3u
+ 4u^2v^2 - 12tv^2 - 2u^3v + 8vut - u^4 + 4u^2t - 2v^4t + 4v^3ut - 4u^3vt + 2u^4t + 2n8 + 4n8t,
-15v^3 + 13v^2u + 3u^2v - u^3 - 3v^5 + 5uv^4 + 2v^3u^2 - 14v^3t - 6u^3v^2 + 26v^2ut + u^4v
- 10vu^2t + u^5 - 2u^3t - 2v^5t + 6v^4ut - 4u^2tv^3 - 4u^3tv^2 + 6u^4tv - 2u^5t - 4n9 - 8n9t
+ 4n8n9, 1 + v^2 - u^2 + 2n7 - 2t, -1 - 2t + v^2 - u^2 + 2n6, vt - ut + n5 - n9 + v - u,
2v^2 - 2uv + tv^2 - 2vut + u^2t - n8 + n4, v - u - 2n9 + 4vt - 4ut - v^3 + v^2u + u^2v - u^3 + 2n3,

```

$$1 + 3v^2 - 2uv - u^2 + tv^2 - 2vut + u^2t - n8 + n2, n13 - v, -t + n11 - 1,$$

$$2 + 9v^2 - 6uv - 3u^2 - 2n8 + v^4 - 2v^3u + 2u^3v - u^4 + 2n1]$$

Since $(8v^2-1)$ cannot be 0, get a new relation.

```
> newrel1 := -t*v^2 - 2*v^2 + 2*u*v + 2*v*u*t - 1 - 2*t - u^2*t + n8; newrel1 :=
```

$$-2*v^2 - t*v^2 + 2*u*v + 2*v*u*t + n8 - 2*t - 1 - u^2*t$$

```
> factor(Basis('union'(allNrels, {dio1, newrel1}), plex(n1, n11, n13, n2, n3, n4, n5, n6, n7, n8, n9, t, u, v)));
```

$$[2u^2t - u^2 - 4uv - 3v^2 - 2tv^2 + 2t + 1 + 4t^2, v(-2v - vt + n9 + ut), u(-2v - vt + n9 + ut), u^2 - 3v^4 - u^4 - 15v^2 + 12vut + 2v^3u - 14tv^2 - 2u^3v + 2u^2t - 2uv + 4n9^2 + 2u^4t + 4u^2v^2 - 2v^4t + 4v^3ut - 4u^3vt, -2v^2 - tv^2 + 2uv + 2vut + n8 - 2t - 1 - u^2t, 1 + v^2 - u^2 + 2n7 - 2t - 1 - 2t + v^2 - u^2 + 2n6, vt - ut + n5 - n9 + v - u, n4 - 1 - 2t,$$

$$v - u - 2n9 + 4vt - 4ut - v^3 + v^2u + u^2v - u^3 + 2n3, -2t + v^2 - u^2 + n2, n13 - v, -t + n11 - 1,$$

$$5v^2 - 2uv - 3u^2 - 4t + v^4 - 2v^3u - 2tv^2 + 2u^3v + 4vut - u^4 - 2u^2t + 2n1]$$

```
> goodrel1 := -v*t - 2*v + n9 + u*t;
```

$$goodrel1 := -2v - vt + n9 + ut$$

Either $goodrel1=0$ or both u and v are zero, by the second and third equations.

```
> badrels1 := {u, v};
```

$$badrels1 := \{u, v\}$$

No integer solutions for t with $badrels1$.

```
> factor(Basis('union'(allNrels, {dio1, newrel1}, badrels1), plex(n1, n11, n13, n2, n3, n4, n5, n6, n7, n8, n9, t, u, v)));
```

$$[v, u, 4t^2 + 2t + 1, n9^2, n8 - 1 - 2t, 2n7 + 1 - 2t, 2n6 - 1 - 2t, -n9 + n5, n4 - 1 - 2t, n3 - n9, n2 - 2t, n13, -t + n11 - 1, n1 - 2t]$$

```
> finalNs := factor(Basis('union'(allNrels, {dio1, newrel1, goodrel1}), plex(n1, n11, n13, n2, n3, n4, n5, n6, n7, n8, n9, t, u, v)));
```

$$finalNs := [2u^2t - u^2 - 4uv - 3v^2 - 2tv^2 + 2t + 1 + 4t^2, -2v - vt + n9 + ut, -2v^2 - tv^2 + 2uv + 2vut + n8 - 2t - 1 - u^2t, 1 + v^2 - u^2 + 2n7 - 2t, -1 - 2t + v^2 - u^2 + 2n6, -v - u + n5, n4 - 1 - 2t, -3v - u + 2vt - 2ut - v^3 + v^2u + u^2v - u^3 + 2n3, -2t + v^2 - u^2 + n2, n13 - v, -t + n11 - 1,$$

$$5v^2 - 2uv - 3u^2 - 4t + v^4 - 2v^3u - 2tv^2 + 2u^3v + 4vut - u^4 - 2u^2t + 2n1]$$

We obtain one diophantine equation in t, u, v and the remaining fusion coefficients can be expressed uniquely as polynomials in t, u, v .

```
> rules1 := 'union'({n10=t, n12=v, n14=u}, solve({seq(finalNs[i], i=2..nops(finalNs)), {n1, n11, n13, n2, n3, n4, n5, n6, n7, n8, n9})));
```

$$rules1 := \{n1 = -\frac{5}{2}v^2 + uv + \frac{3}{2}u^2 + 2t - \frac{1}{2}v^4 + v^3u + tv^2 - u^3v - 2vut + \frac{1}{2}u^4 + u^2t, n10 = t,$$

$$n11 = 1 + t, n12 = v, n13 = v, n14 = u, n2 = 2t - v^2 + u^2,$$

$$n3 = \frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3, n4 = 2t + 1, n5 = u + v, n6 = \frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2},$$

$$n7 = -\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}, n8 = tv^2 + 2v^2 - 2uv - 2vut + 1 + 2t + u^2t, n9 = vt + 2v - ut$$

The anticipated unique solution corresponding to the known example of SU(3)₄/Z₃.

> subs({t=0, v=0, u=1}, %);

$$\{n1 = 2, n10 = 0, n11 = 1, n12 = 0, n13 = 0, n14 = 1, n2 = 1, n3 = 1, n4 = 1, n5 = 1, n6 = 1, n7 = 0, n8 = 1, n9 = 0\}$$

The fusion matrices with all but t,u,v eliminated.

> M1s:=subs(rules1,M1); M2s:=subs(rules1,M2); M3s:=subs(rules1,M3);

M1s:=

$$[0, 1, 0, 0, 0]$$

$$\left[1, -\frac{5}{2}v^2 + uv + \frac{3}{2}u^2 + 2t - \frac{1}{2}v^4 + v^3u + tv^2 - u^3v - 2vut + \frac{1}{2}u^4 + u^2t, 2t - v^2 + u^2, \right.$$

$$\left. \frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3, \frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3 \right]$$

]

$$[0, 2t - v^2 + u^2, 2t + 1, u + v, u + v]$$

$$\left[0, \frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3, u + v, \frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}, -\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right]$$

$$\left[0, \frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3, u + v, -\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}, \frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right]$$

$$M2s := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2t - v^2 + u^2 & 2t + 1 & u + v & u + v \\ 1 & 2t + 1 & tv^2 + 2v^2 - 2uv - 2vut + 1 + 2t + u^2t & vt + 2v - ut & vt + 2v - ut \\ 0 & u + v & vt + 2v - ut & t & 1 + t \\ 0 & u + v & vt + 2v - ut & 1 + t & t \end{bmatrix}$$

M3s:=

$$[0, 0, 0, 0, 1]$$

$$\left[0, \frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3, u + v, -\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}, \frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right]$$

$$[0, u + v, vt + 2v - ut, 1 + t, t]$$

$$\left[1, \frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}, t, v, v \right]$$

$$\left[0, -\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}, 1 + t, u, v \right]$$

The unique positive character is psi0.

> psi0:=Vector([1,d,f,g,g]):

> psirels:='minus'('union'(convert(evalm(M1s*psi0-d*psi0),set),convert(evalm(M2s*psi0-f*psi0),set),convert(evalm(M3s*psi0-g*psi0),set)),{0}));

psirels := {(2t-v^2+u^2)d+(2t+1)f+2(u+v)g-df,

$$1+(2t+1)d+(tv^2+2v^2-2uv-2vut+1+2t+u^2t)f+2(vt+2v-ut)g-f^2,$$

$$1+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+tf+2vg-g^2, 1$$

$$+\left(-\frac{5}{2}v^2+uv+\frac{3}{2}u^2+2t-\frac{1}{2}v^4+v^3u+tv^2-u^3v-2vut+\frac{1}{2}u^4+u^2t\right)d+(2t-v^2+u^2)f$$

$$+2\left(\frac{3}{2}v+\frac{1}{2}u-vt+ut+\frac{1}{2}v^3-\frac{1}{2}v^2u-\frac{1}{2}u^2v+\frac{1}{2}u^3\right)g-d^2,$$

$$(u+v)d+(vt+2v-ut)f+tg+(1+t)g-fg,\left(-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+(1+t)f+ug+vg-g^2,$$

$$\left(\frac{3}{2}v+\frac{1}{2}u-vt+ut+\frac{1}{2}v^3-\frac{1}{2}v^2u-\frac{1}{2}u^2v+\frac{1}{2}u^3\right)d+(u+v)f+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)g$$

$$+\left(-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)g-dg\}$$

We combine the nontrivial relations obtained so far.

> SandNrels:='union'(psirels,{diol,orthrel});

SandNrels := {f-d+df-2g^2,(2t-v^2+u^2)d+(2t+1)f+2(u+v)g-df,

$$1+(2t+1)d+(tv^2+2v^2-2uv-2vut+1+2t+u^2t)f+2(vt+2v-ut)g-f^2,$$

$$1+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+tf+2vg-g^2, 1$$

$$+\left(-\frac{5}{2}v^2+uv+\frac{3}{2}u^2+2t-\frac{1}{2}v^4+v^3u+tv^2-u^3v-2vut+\frac{1}{2}u^4+u^2t\right)d+(2t-v^2+u^2)f$$

$$+2\left(\frac{3}{2}v+\frac{1}{2}u-vt+ut+\frac{1}{2}v^3-\frac{1}{2}v^2u-\frac{1}{2}u^2v+\frac{1}{2}u^3\right)g-d^2,$$

$$(u+v)d+(vt+2v-ut)f+tg+(1+t)g-fg,\left(-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+(1+t)f+ug+vg-g^2,$$

$$\left(\frac{3}{2}v+\frac{1}{2}u-vt+ut+\frac{1}{2}v^3-\frac{1}{2}v^2u-\frac{1}{2}u^2v+\frac{1}{2}u^3\right)d+(u+v)f+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)g$$

$$\left. + \left(-\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right) g - d g, 2 u^2 t - u^2 - 4 u v - 3 v^2 - 2 t v^2 + 2 t + 1 + 4 t^2 \right\}$$

> SandNbasis := Basis(SandNrels, tdeg(g, f, d, u, v, t));

SandNbasis := [u g - 1 - v g + f - d, d - d f - f + 2 g^2,
 2 u^2 t - u^2 - 4 u v - 3 v^2 - 2 t v^2 + 2 t + 1 + 4 t^2, f u t - f v t + f g - d u - 2 f v - d v - 2 t g - g, 8 g v^2
 - 2 d u t + 4 f v t - 2 d v t - 4 g t^2 - 2 f g - f u + 3 d u - f v + 7 d v + 2 t g - 2 u t - 2 v t + g + u + 5 v,
 4 d u v - 2 + 4 d v^2 - 2 f d t + 8 g v t + 4 f t^2 + d f - 4 v g - 4 t f + f - 3 d + 4 t,
 2 f d v - 2 g d t - d u t + d v t - 2 f g + d g - f u + d u + f v + d v + g,
 d u^2 + 2 - d v^2 - d f + 4 v g + 2 t f + 2 d t - f + 2 d,
 f u^2 + 4 g f v - f v^2 - 2 f d t - 2 f^2 + d f + 2 t f + 2 f - d,
 2 f d u - 2 g d t - d u t + d v t + 2 f g - 3 d g + f u - d u - f v + 3 d v - 3 g,
 f^2 d - 4 g f v - 4 g d v - 2 f^2 t + f^2 - 2 d^2 + 2 d t - f - d,
 g f d - d^2 u - d^2 v - 2 g f t - 2 g d t - f g - f u - f v, 8 d t v^2 - 2 f d t^2 + 8 g v t^2 + 4 f t^3 - 8 g d v
 + 8 d v^2 + f d t + 4 d^2 t - 4 g v t - 4 f t^2 - 4 d t^2 + 3 d f - 2 d^2 - 4 v g - t f + d t + 4 t^2 + f - d - 2 t,
 8 g f v t - 4 f d t^2 - 4 g f v + 4 f u v + 4 f v^2 - 4 f^2 t + 4 f d t + 2 f^2 - d f - 2 d t - 3 f + d, -4 d g - 9 f g
 + 16 t g + 16 g f t^2 + 8 g d t^2 - 8 f v t^2 + 4 g f^2 + f^2 u - 7 f^2 v - 4 g f t - 4 g d t - 8 g t^2 + 8 f^2 v t^2
 - 8 g f t^3 - 8 f u v^2 - 8 f v^3 - 4 g f^2 t + 5 d u + 13 d v + 10 f v - 2 f u - 6 d v t - 6 d u t + 16 f v t
 + 4 d u t^2 - 4 d v t^2]

The next step is to introduce the relations involving the roots of unity theta_i. The first goal is to show that th2 satisfies a nontrivial degree 3 (or less) polynomial in Q[d].

> psith0 := Vector([1, d*th1, f*th2, g*th3, g*th3]);

> Thetarels := simplify(subs({h1=1/2*(f-d-1)}, [th1^2*S[2,2]-(sum('M1[k,2]*psith0[k]',k=1..5)), th1*th2*S[2,3]-(sum('M1[k,3]*psith0[k]',k=1..5)), th1*th3*S[2,4]-(sum('M1[k,4]*psith0[k]',k=1..5)), th3*th2*S[4,3]-(sum('M2[k,4]*psith0[k]',k=1..5)), th2^2*S[3,3]-(sum('M2[k,3]*psith0[k]',k=1..5)), th3^2*(S[4,4]+S[4,5])-(sum('M3[k,4]+M3[k,5])*psith0[k]',k=1..5))));

Thetarels := [-th1^2 f - 1 - n1 d th1 - n2 f th2 - 2 n3 g th3,
 -th1 th2 - n2 d th1 - n4 f th2 - 2 n5 g th3, th1 th3 g - n3 d th1 - n5 f th2 - n6 g th3 - n7 g th3,
 -th3 th2 g - n5 d th1 - n9 f th2 - n10 g th3 - n11 g th3, th2^2 d - 1 - n4 d th1 - n8 f th2 - 2 n9 g th3,
 th3^2 f - th3^2 d - th3^2 - 1 - d th1 n6 - d th1 n7 - f th2 n11 - f th2 n10 - 2 g th3 n12 - g th3 n13
 - g th3 n14]

We solve for th1 first, note that the denominator is not 0.

> solve(Thetarels[2], th1);

$$-\frac{n4 f th2 + 2 n5 g th3}{th2 + n2 d}$$

We substitute back in to eliminate th1 and take at the numerators of the resulting rational functions, which give us our new relations.

```
> Thetarels2:=factor( numer( subs( { th1=solve( Thetarels[2], th1) }, Thetarels) ) );
```

```
Thetarels2 := [-n4^2 f^3 th2^2 - 4 n4 f^2 th2 n5 g th3 - 4 f n5^2 g^2 th3^2 - th2^2 - 2 th2 n2 d - n2^2 d^2
+ n1 d n4 f th2^2 + n1 d^2 n4 f th2 n2 + 2 n1 d n5 g th3 th2 + 2 n1 d^2 n5 g th3 n2 - n2 f th2^3
- 2 n2^2 f th2^2 d - n2^3 f th2 d^2 - 2 n3 g th3 th2^2 - 4 n3 g th3 th2 n2 d - 2 n3 g th3 n2^2 d^2, 0,
- g th3 n4 f th2 - 2 g^2 th3^2 n5 + n3 d n4 f th2 + 2 n3 d n5 g th3 - n5 f th2^2 - n5 f th2 n2 d
- n6 g th3 th2 - n6 g th3 n2 d - n7 g th3 th2 - n7 g th3 n2 d, -th3 th2^2 g - th3 th2 g n2 d
+ n5 d n4 f th2 + 2 n5^2 d g th3 - n9 f th2^2 - n9 f th2 n2 d - n10 g th3 th2 - n10 g th3 n2 d
- n11 g th3 th2 - n11 g th3 n2 d, th2^3 d + th2^2 d^2 n2 - th2 - n2 d + n4^2 d f th2 + 2 n4 d n5 g th3
- n8 f th2^2 - n8 f th2 n2 d - 2 n9 g th3 th2 - 2 n9 g th3 n2 d, th3^2 f th2 + th3^2 f n2 d - th3^2 d th2
- th3^2 d^2 n2 - th3^2 th2 - th3^2 n2 d - th2 - n2 d + d n6 n4 f th2 + 2 d n6 n5 g th3 + d n7 n4 f th2
+ 2 d n7 n5 g th3 - f th2^2 n11 - f th2 n11 n2 d - f th2^2 n10 - f th2 n10 n2 d - 2 g th3 n12 th2
- 2 g th3 n12 n2 d - g th3 n13 th2 - g th3 n13 n2 d - g th3 n14 th2 - g th3 n14 n2 d]
```

```
> map( degree, Thetarels2, th3 );
```

```
[2, -∞, 2, 1, 1, 2]
```

Notice: either the denominator is non-zero, or th2 satisfies a degree 2 polynomial.

```
> solve( Thetarels2[4], th3 );
```

$$-\frac{f th2 (n9 th2 + n9 n2 d - n5 d n4)}{g (-2 n5^2 d + th2^2 + th2 n2 d + n10 th2 + n10 n2 d + n11 th2 + n11 n2 d)}$$

```
> Thetarels3:=factor( numer( subs( { th3=solve( Thetarels2[4], th3) }, Thetarels2) ) );
```

```
Thetarels3 := [(th2 + n2 d)^2 (4 n4 f^3 th2^3 n5 n9 - 2 n4^2 f^3 th2^2 n10 n11 - 2 n2 f th2^3 n10 n11
+ 4 d n2 f th2^2 n5^2 n11 + 4 d n2 n3 f th2^3 n9 - 4 d n2^2 f th2^2 n10 n11 + 4 n5^2 d th2^2 - 2 th2^3 n2 d
- th2^2 n2^2 d^2 - 2 n10 th2^2 n11 - n10^2 n2^2 d^2 - n11^2 n2^2 d^2 + 4 n5^2 d^2 th2 n2 + 4 n5^2 d n10 th2
+ 4 n5^2 d^2 n10 n2 + 4 n5^2 d n11 th2 + 4 n5^2 d^2 n11 n2 - 4 th2^2 n10 n2 d - 4 th2^2 n11 n2 d
- 2 th2 n2^2 d^2 n10 - 2 th2 n2^2 d^2 n11 - 2 n10^2 th2 n2 d - 2 n10 n2^2 d^2 n11 - 2 n11^2 th2 n2 d
- n4^2 f^3 th2^4 + 4 n4 f^3 th2^2 n5 n9 n10 + 4 n4 f^3 th2^2 n5 n9 n11 - 2 d^2 n1 n4 f th2^2 n5^2
+ 4 d^2 n1 n5^3 f th2 n9 + 4 d^2 n3 f th2 n5^3 n4 - 2 d^2 n2^3 f th2 n10 n11 + 4 d^2 n2^2 f th2 n5^2 n10
+ 4 d^2 n2^2 f th2 n5^2 n11 + 2 d^2 n2^2 n3 f th2^2 n9 + d^2 n2 n1 n4 f th2^3 + d n1 n4 f th2^2 n10^2
+ d n1 n4 f th2^2 n11^2 + 2 d n1 n4 f th2^3 n10 + 2 d n1 n4 f th2^3 n11 - 2 d n1 n5 f th2^3 n9
- 4 d n3 f th2^2 n9 n5^2 - 2 d n3 f th2^3 n5 n4 + 4 d n2 f th2^2 n5^2 n10 - 2 d^2 n1 n4 f th2 n5^2 n11
+ 2 d^2 n2^2 n3 f th2 n9 n11 + 2 d^2 n2^2 n3 f th2 n9 n10 + 2 d^2 n2 n1 n4 f th2^2 n10
- 2 d^2 n2 n1 n5 f th2^2 n9 + d^2 n2 n1 n4 f th2 n11^2 - 4 d^2 n2 n3 f th2 n9 n5^2
+ 2 d^2 n2 n1 n4 f th2^2 n11 - 2 d^2 n2 n3 f th2^2 n5 n4 + d^2 n2 n1 n4 f th2 n10^2
+ 2 d n1 n4 f th2^2 n10 n11 - 2 d n1 n5 f th2^2 n9 n10 - 2 d n1 n5 f th2^2 n9 n11]
```

$$\begin{aligned}
& -2 d n^3 f th^2 n^5 n^4 n^{10} - 2 d n^3 f th^2 n^5 n^4 n^{11} + 4 d n^2 n^3 f th^2 n^9 n^{11} \\
& + 4 d n^2 n^3 f th^2 n^9 n^{10} + 2 n^3 f th^2 n^9 n^{10} + 2 n^3 f th^2 n^9 n^{11} - th^4 - 4 n^{10} th^2 n^{11} n^2 d \\
& - n^4 f^3 th^2 n^{10} - n^4 f^3 th^2 n^{11} - 2 n^4 f^3 th^2 n^{10} - 2 n^4 f^3 th^2 n^{11} - 4 f^3 n^5 th^2 n^9 \\
& + 2 n^3 f th^2 n^9 - 2 n^2 f th^2 n^{11} - n^2 f th^2 n^{10} - n^2 f th^2 n^{11} - 2 n^2 f th^2 n^{10} \\
& - 2 d^2 n^2 f th^2 n^{11} - d^2 n^2 f th^2 n^{11} - 2 d^2 n^2 f th^2 n^{10} - d^2 n^2 f th^2 n^{10} \\
& + 4 d^2 n^2 f th^2 n^5 - 4 d^2 n^2 f th^2 n^5 + d n^1 n^4 f th^2 - 2 d n^2 f th^2 n^{11} - 2 d n^2 f th^2 n^{10} \\
& - 4 d n^2 f th^2 n^{10} - 4 d n^2 f th^2 n^{11} + 4 d n^2 f th^2 n^5 - 2 d^2 n^1 n^4 f th^2 n^5 n^{10} \\
& - 2 d^2 n^2 n^1 n^5 f th^2 n^9 n^{11} - 2 d^2 n^2 n^3 f th^2 n^5 n^4 n^{10} - 2 d^2 n^2 n^3 f th^2 n^5 n^4 n^{11} \\
& - 2 d^2 n^2 n^1 n^5 f th^2 n^9 n^{10} + 2 d^2 n^2 n^1 n^4 f th^2 n^{10} n^{11} - 4 n^5 d^2 - 2 th^2 n^{10} - 2 th^2 n^{11} \\
& - n^{10} th^2 - n^{11} th^2 - 2 n^2 f th^2 d - n^2 f th^2 - n^2 f th^2 d^2, 0, -f th^2 (th^2 + n^2 d) (\\
& 2 n^5 th^2 n^{10} n^{11} + d^2 n^2 n^5 n^{10} + d^2 n^2 n^5 n^{11} - 4 d^2 n^2 n^5 n^{11} - 4 d^2 n^2 n^5 n^{10} \\
& - 4 d n^5 th^2 n^{11} - 4 d n^5 th^2 n^{10} - d n^2 n^4 f th^2 n^9 n^{10} - d n^2 n^4 f th^2 n^9 n^{11} \\
& + d^2 n^2 n^6 n^5 n^4 n^{11} - 2 d^2 n^2 n^3 n^4 th^2 n^{10} - 2 d^2 n^2 n^3 n^4 th^2 n^{11} + 2 d^2 n^2 n^3 n^5 n^9 n^{11} \\
& + d^2 n^2 n^6 n^5 n^4 n^{10} + 2 d^2 n^2 n^3 n^5 n^9 th^2 + d^2 n^2 n^7 n^5 n^4 th^2 + d^2 n^2 n^6 n^5 n^4 th^2 \\
& + d^2 n^2 n^7 n^5 n^4 n^{11} + 2 d^2 n^2 n^3 n^5 n^9 n^{10} - 2 d^2 n^2 n^3 n^4 n^{10} n^{11} + d^2 n^2 n^7 n^5 n^4 n^{10} \\
& - 2 d n^2 n^7 th^2 n^9 n^{10} - 2 d n^2 n^7 th^2 n^9 n^{11} + 2 d n^2 n^5 f th^2 n^9 - 2 d n^2 n^6 th^2 n^9 n^{10} \\
& - 2 d n^2 n^6 th^2 n^9 n^{11} + d n^7 th^2 n^5 n^4 n^{11} - 2 d n^4 f th^2 n^9 n^5 + d n^6 th^2 n^5 n^4 n^{10} \\
& + d n^6 th^2 n^5 n^4 n^{11} + d n^7 th^2 n^5 n^4 n^{10} - d n^2 n^4 f th^2 n^9 + d n^4 f th^2 n^5 n^{10} \\
& + d n^4 f th^2 n^5 n^{11} - 2 d n^3 n^4 n^{10} th^2 n^{11} + 2 d n^3 n^5 n^9 th^2 n^{10} + 2 d n^3 n^5 n^9 th^2 n^{11} \\
& + n^5 th^4 - 4 d^2 n^5 th^2 n^2 + d^2 n^5 th^2 n^2 + 2 d n^5 th^2 n^2 + n^5 th^2 n^{11} + n^5 th^2 n^{10} \\
& - n^7 th^2 n^9 - n^6 th^2 n^9 + 2 n^5 th^2 n^{10} + 4 d^2 n^5 + 2 d^2 n^5 th^2 n^2 n^{11} + 4 d n^5 th^2 n^{10} n^2 \\
& + 4 d n^5 th^2 n^{11} n^2 + 2 d^2 n^5 th^2 n^2 n^{10} + 4 d n^2 n^5 th^2 n^{10} n^{11} + 2 d^2 n^2 n^5 n^{10} n^{11} \\
& + 2 d n^2 n^5 th^2 n^{11} + 2 d n^2 n^5 th^2 n^{10} - n^4 f th^2 n^9 + 2 n^5 f th^2 n^9 - n^6 th^2 n^9 n^{10} \\
& - n^6 th^2 n^9 n^{11} - n^7 th^2 n^9 n^{10} - n^7 th^2 n^9 n^{11} - 2 d^2 n^6 n^5 n^4 - 2 d^2 n^7 n^5 n^4 \\
& - 4 d^2 n^3 n^5 n^9 - d n^3 n^4 th^2 - 4 d n^5 th^2 + 2 n^5 th^2 n^{11} - n^4 f th^2 n^9 n^{10} \\
& - n^4 f th^2 n^9 n^{11} + 2 d^2 n^2 n^5 n^6 n^9 + 2 d^2 n^2 n^5 n^7 n^9 - d^2 n^2 n^{10} n^7 n^9 \\
& + 2 d^2 n^3 n^4 n^5 n^{10} + 2 d^2 n^3 n^4 n^5 n^{11} - d^2 n^2 n^{11} n^7 n^9 - d^2 n^2 n^6 n^9 n^{10} \\
& - d^2 n^2 n^7 n^9 th^2 - d^2 n^2 n^6 n^9 th^2 - d^2 n^2 n^6 n^9 n^{11} - d^2 n^2 n^3 n^4 n^{11} - d^2 n^2 n^3 n^4 th^2 \\
& - d^2 n^2 n^3 n^4 n^{10} + 2 d^2 n^3 n^4 n^5 th^2 + d n^4 f th^2 n^5 - 2 d n^3 n^4 th^2 n^{10} \\
& - 2 d n^3 n^4 th^2 n^{11} - d n^3 n^4 n^{10} th^2 - d n^3 n^4 n^{11} th^2 + 2 d n^3 n^5 n^9 th^2 \\
& + 2 d n^6 th^2 n^9 n^5 + d n^6 th^2 n^5 n^4 + 2 d n^7 th^2 n^9 n^5 + d n^7 th^2 n^5 n^4 - 2 d n^2 n^7 th^2 n^9 \\
& - 2 d n^2 n^6 th^2 n^9), 0, (th^2 + n^2 d) (2 n^5 d - n^{11} th^2 - n^{10} n^2 d - n^{11} n^2 d - th^2 n^2 d
\end{aligned}$$

$$\begin{aligned}
& + th^3 d^2 n_2 - d n_2 n_8 f th_2 n_{10} - d n_2 n_8 f th_2 n_{11} - 4 d n_5 n_4 f th_2 n_9 + 2 d n_2 n_9^2 f th_2 \\
& - d n_2 n_8 f th_2^2 + 2 d n_8 f th_2 n_5^2 + d n_4^2 f th_2 n_{10} + d n_4^2 f th_2 n_{11} + th_2^4 d - n_8 f th_2^2 n_{10} \\
& - n_8 f th_2^2 n_{11} + d^2 th_2^2 n_{10} n_2 + d^2 th_2^2 n_{11} n_2 + d n_4^2 f th_2^2 - 2 d^2 n_5^2 th_2^2 - th_2^2 - n_8 f th_2^3 \\
& + 2 n_9^2 f th_2^2 + d th_2^3 n_{11} + d th_2^3 n_{10} - n_{10} th_2), -(th_2 + n_2 d) (2 th_2^2 n_{11} g^2 n_{10} \\
& + 2 th_2^3 n_{11} g^2 + th_2^2 n_{11}^2 g^2 + 2 th_2^3 n_{10} g^2 + th_2^2 n_{10}^2 g^2 - 2 d^2 n_2^2 f th_2^2 n_{12} g^2 n_9 \\
& - d^2 n_2^2 f th_2^2 n_{14} g^2 n_9 + 3 d^2 n_2^2 f th_2 n_{11} g^2 n_{10}^2 + 4 d^2 n_2^2 f th_2^2 n_{11} g^2 n_{10} \\
& + 3 d^2 n_2^2 f th_2 n_{11}^2 g^2 n_{10} + 2 d^2 n_2 f^3 th_2^2 n_9 n_5 n_4 - 4 d^2 n_2 f th_2^2 n_{11} g^2 n_5^2 \\
& - 4 d^2 n_2 f th_2 n_{11}^2 g^2 n_5^2 - 4 d^2 n_2 f th_2^2 n_{10} g^2 n_5^2 - 4 d^2 n_2 f th_2 n_{10}^2 g^2 n_5^2 \\
& - d^2 n_2 n_6 n_4 f th_2^3 g^2 - d^2 n_2 n_7 n_4 f th_2^3 g^2 + d f th_2^3 n_{14} g^2 n_5 n_4 - 2 d n_6 n_4 f th_2^3 g^2 n_{10} \\
& - 2 d n_6 n_4 f th_2^3 g^2 n_{11} - d n_6 n_4 f th_2^2 g^2 n_{10}^2 - d n_6 n_4 f th_2^2 g^2 n_{11}^2 + 2 d n_6 n_5 f th_2^3 g^2 n_9 \\
& - 4 d n_2 f th_2^3 n_{12} g^2 n_9 - 2 d n_2 f th_2^3 n_{13} g^2 n_9 - 2 d n_2 f th_2^3 n_{14} g^2 n_9 \\
& + 6 d n_2 f th_2^2 n_{11} g^2 n_{10}^2 + 2 d^2 n_7 n_4 f th_2 g^2 n_5^2 n_{10} + 2 d^2 n_7 n_4 f th_2 g^2 n_5^2 n_{11} \\
& + 2 d^2 n_6 n_4 f th_2 g^2 n_5^2 n_{10} + 2 d^2 n_6 n_4 f th_2 g^2 n_5^2 n_{11} - 2 d^2 n_2^2 f th_2 n_{12} g^2 n_9 n_{10} \\
& - 2 d^2 n_2^2 f th_2 n_{12} g^2 n_9 n_{11} - d^2 n_2^2 f th_2 n_{13} g^2 n_9 n_{10} - d^2 n_2^2 f th_2 n_{13} g^2 n_9 n_{11} \\
& - d^2 n_2^2 f th_2 n_{14} g^2 n_9 n_{10} - d^2 n_2^2 f th_2 n_{14} g^2 n_9 n_{11} - 8 d^2 n_2 f th_2 n_{11} g^2 n_5^2 n_{10} \\
& + 2 d^2 n_2 n_6 n_5 f th_2^2 g^2 n_9 - d^2 n_2 n_7 n_4 f th_2 g^2 n_{10}^2 - d^2 n_2 n_7 n_4 f th_2 g^2 n_{11}^2 \\
& - 2 d^2 n_2 n_7 n_4 f th_2^2 g^2 n_{10} - 2 d^2 n_2 n_7 n_4 f th_2^2 g^2 n_{11} + 2 d^2 n_2 n_7 n_5 f th_2^2 g^2 n_9 \\
& + 4 d^2 n_2 f th_2 n_{12} g^2 n_9 n_5^2 + 2 d^2 n_2 f th_2^2 n_{12} g^2 n_5 n_4 + 2 d^2 n_2 f th_2 n_{13} g^2 n_9 n_5^2 \\
& + d^2 n_2 f th_2^2 n_{13} g^2 n_5 n_4 + 2 d^2 n_2 f th_2 n_{14} g^2 n_9 n_5^2 + d^2 n_2 f th_2^2 n_{14} g^2 n_5 n_4 \\
& - d^2 n_2 n_6 n_4 f th_2 g^2 n_{11}^2 - d^2 n_2 n_6 n_4 f th_2 g^2 n_{10}^2 - 2 d^2 n_2 n_6 n_4 f th_2^2 g^2 n_{10} \\
& - 2 d^2 n_2 n_6 n_4 f th_2^2 g^2 n_{11} - 2 d^2 n_2 n_6 n_4 f th_2 g^2 n_{10} n_{11} + 2 d^2 n_2 n_6 n_5 f th_2 g^2 n_9 n_{10} \\
& + 2 d^2 n_2 n_6 n_5 f th_2 g^2 n_9 n_{11} - 2 d^2 n_2 n_7 n_4 f th_2 g^2 n_{10} n_{11} + 2 d^2 n_2 n_7 n_5 f th_2 g^2 n_9 n_{10} \\
& + 2 d^2 n_2 n_7 n_5 f th_2 g^2 n_9 n_{11} + 2 d^2 n_2 f th_2 n_{12} g^2 n_5 n_4 n_{10} + 2 d^2 n_2 f th_2 n_{12} g^2 n_5 n_4 n_{11} \\
& + d^2 n_2 f th_2 n_{13} g^2 n_5 n_4 n_{10} + d^2 n_2 f th_2 n_{13} g^2 n_5 n_4 n_{11} + d^2 n_2 f th_2 n_{14} g^2 n_5 n_4 n_{10} \\
& + d^2 n_2 f th_2 n_{14} g^2 n_5 n_4 n_{11} - 2 d n_6 n_4 f th_2^2 g^2 n_{10} n_{11} + 2 d n_6 n_5 f th_2^2 g^2 n_9 n_{10} \\
& + 2 d n_6 n_5 f th_2^2 g^2 n_9 n_{11} + 2 d n_7 n_5 f th_2^2 g^2 n_9 n_{10} + 2 d n_7 n_5 f th_2^2 g^2 n_9 n_{11} \\
& - 2 d n_7 n_4 f th_2^2 g^2 n_{10} n_{11} + 2 d f th_2^2 n_{12} g^2 n_5 n_4 n_{10} + 2 d f th_2^2 n_{12} g^2 n_5 n_4 n_{11} \\
& + d f th_2^2 n_{13} g^2 n_5 n_4 n_{10} + d f th_2^2 n_{13} g^2 n_5 n_4 n_{11} + d f th_2^2 n_{14} g^2 n_5 n_4 n_{10} \\
& + d f th_2^2 n_{14} g^2 n_5 n_4 n_{11} - 2 d n_2 f th_2^2 n_{14} g^2 n_9 n_{10} - 2 d n_2 f th_2^2 n_{14} g^2 n_9 n_{11} \\
& - 4 d n_2 f th_2^2 n_{12} g^2 n_9 n_{10} - 4 d n_2 f th_2^2 n_{12} g^2 n_9 n_{11} - 2 d n_2 f th_2^2 n_{13} g^2 n_9 n_{10} \\
& - 2 d n_2 f th_2^2 n_{13} g^2 n_9 n_{11} - 2 d^3 f^2 th_2^2 n_9 n_5 n_4 n_2 - d^2 n_2^2 f th_2^2 n_{13} g^2 n_9 \\
& - 2 d^2 f^2 th_2^2 n_9 n_5 n_4 n_2 - 4 d^2 n_6 n_5^3 f th_2 g^2 n_9 + 2 d^2 n_7 n_4 f th_2^2 g^2 n_5^2
\end{aligned}$$

$$\begin{aligned}
& -4 d^2 n7 n5^3 f th2 g^2 n9 - 4 d^2 f th2 n12 g^2 n5^3 n4 - 2 d^2 f th2 n13 g^2 n5^3 n4 \\
& - 2 d^2 f th2 n14 g^2 n5^3 n4 + 2 d^2 n6 n4 f th2^2 g^2 n5^2 - 2 d^2 f^2 th2^3 n9 n5 n4 + f^2 th2^4 n9^2 \\
& - f^3 th2^4 n9^2 + 4 d^2 n5^4 g^2 - 2 f th2^4 n12 g^2 n9 - f th2^4 n13 g^2 n9 - f th2^4 n14 g^2 n9 \\
& + 4 f th2^4 n11 g^2 n10 + 3 f th2^3 n11 g^2 n10^2 + 3 f th2^3 n11^2 g^2 n10 + d^3 f^2 th2^2 n9^2 n2^2 \\
& + d^3 f^2 th2^2 n5^2 n4^2 - d^2 n2^2 f^3 th2^2 n9^2 + 2 d^2 n2^2 n11 g^2 n10 + 2 d^2 n2^2 g^2 th2 n10 \\
& + d^2 f^2 th2^2 n5^2 n4^2 + d^2 f^2 th2^2 n9^2 n2^2 - d^2 f^3 th2^2 n5^2 n4^2 - 4 d^2 n2 g^2 n5^2 n10 \\
& - 4 d^2 n2 g^2 n5^2 n11 + 2 d^2 n2 f^2 th2^3 n9^2 - 4 d^2 n2 g^2 n5^2 th2 - 4 d th2 g^2 n5^2 n10 \\
& - 4 d th2 g^2 n5^2 n11 + 4 d n2 th2^2 n10 g^2 + 4 d n2 th2^2 n11 g^2 + 2 d n2 th2 n10^2 g^2 \\
& + 2 d n2 th2 n11^2 g^2 - 2 d n2 f^3 th2^3 n9^2 + 2 d n2 f^2 th2^3 n9^2 + 8 d n2 f th2^3 n11 g^2 n10 \\
& + 6 d n2 f th2^2 n11^2 g^2 n10 - 8 d f th2^2 n11 g^2 n5^2 n10 - 2 d n7 n4 f th2^3 g^2 n10 \\
& - 2 d n7 n4 f th2^3 g^2 n11 - d n7 n4 f th2^2 g^2 n10^2 - d n7 n4 f th2^2 g^2 n11^2 + 2 d n7 n5 f th2^3 g^2 n9 \\
& + 4 d f th2^2 n12 g^2 n9 n5^2 + 2 d f th2^3 n12 g^2 n5 n4 + 2 d f th2^2 n13 g^2 n9 n5^2 \\
& + d f th2^3 n13 g^2 n5 n4 + 2 d f th2^2 n14 g^2 n9 n5^2 - 2 f th2^3 n12 g^2 n9 n10 \\
& - 2 f th2^3 n12 g^2 n9 n11 - f th2^3 n13 g^2 n9 n10 - f th2^3 n13 g^2 n9 n11 - f th2^3 n14 g^2 n9 n10 \\
& - f th2^3 n14 g^2 n9 n11 + d^2 n2^2 f th2 n10^3 g^2 + 4 d^2 f th2 n11 g^2 n5^4 + 4 d^2 f th2 n10 g^2 n5^4 \\
& + 2 d^2 n2^2 f th2^2 n10^2 g^2 + d^2 n2^2 f th2 n11^3 g^2 + 2 d^2 n2^2 f th2^2 n11^2 g^2 + d^2 n2^2 f g^2 th2^3 n11 \\
& + d^2 n2^2 f g^2 th2^3 n10 + 2 d n4 f^3 th2^3 n5 n9 - 2 d f^2 th2^3 n9 n5 n4 - 4 d f th2^3 n11 g^2 n5^2 \\
& - 4 d f th2^2 n11^2 g^2 n5^2 - 4 d f th2^3 n10 g^2 n5^2 - 4 d f th2^2 n10^2 g^2 n5^2 - d n6 n4 f th2^4 g^2 \\
& - d n7 n4 f th2^4 g^2 + 4 d n2 th2 n11 g^2 n10 + 2 d n2 f th2^4 n11 g^2 + 2 d n2 f th2^2 n11^3 g^2 \\
& + 4 d n2 f th2^3 n11^2 g^2 + 2 d n2 f th2^4 n10 g^2 + 2 d n2 f th2^2 n10^3 g^2 + 4 d n2 f th2^3 n10^2 g^2 \\
& + 2 d^2 n2^2 g^2 th2 n11 + f th2^5 n11 g^2 + 2 f th2^4 n11^2 g^2 + f th2^5 n10 g^2 + 2 f th2^4 n10^2 g^2 \\
& + f th2^3 n10^3 g^2 + f th2^3 n11^3 g^2 + d^2 n2^2 n11^2 g^2 + d^2 n2^2 n10^2 g^2 + d^2 n2^2 g^2 th2^2 \\
& + d f^2 th2^4 n9^2 - 4 d th2^2 g^2 n5^2 + 2 d n2 th2^3 g^2 + th2^4 g^2)]
\end{aligned}$$

We remove spurious factors like (th2+n2d)

```

> Thetarels3red:=collect(simplify([Thetarels3[1]/(th2+n2*d)^2, Thetarels3[3]/(f*th2*(th2+n2*d)), Thetarels3[5]/(th2+n2*d), Thetarels3[6]/(th2+n2*d)]), th2);

```

Thetarels3red := [-2 f th2⁵

$$\begin{aligned}
& + (-1 + d n1 n4 f - f^3 n4^2 + 2 n3 f n9 - 2 n2 f n11 - 2 n2^2 f d - 2 n2 f n10) th2^4 + (4 n4 f^3 n5 n9 \\
& - 2 n4^2 f^3 n11 + 2 n3 f n9 n11 + 4 d n2 n3 f n9 - n2 f n10^2 - n2 f n11^2 + 2 n3 f n9 n10 \\
& + d^2 n2 n1 n4 f - 2 n4^2 f^3 n10 - 2 n10 + 2 d n1 n4 f n10 - 2 n2 f n10 n11 - 4 d n2^2 f n10 \\
& - 4 d n2^2 f n11 - 2 d n3 f n5 n4 - n2^3 f d^2 - 2 n2 d - 2 d n1 n5 f n9 + 4 d n2 f n5^2 - 2 n11 \\
& + 2 d n1 n4 f n11) th2^3 + (-n2^2 d^2 + 4 n5^2 d - 4 n10 n2 d - 4 n11 n2 d - n10^2 - n11^2 - 2 n11 n10 \\
& - n4^2 f^3 n10^2 - n4^2 f^3 n11^2 - 4 f^3 n5^2 n9^2 + 2 d^2 n2 n1 n4 f n10 - 2 d^2 n2 n1 n5 f n9
\end{aligned}$$

$$\begin{aligned}
&+ 2 d^2 n_2 n_1 n_4 f n_{11} - 2 d^2 n_2 n_3 f n_5 n_4 + 2 d n_1 n_4 f n_{10} n_{11} - 2 d n_1 n_5 f n_9 n_{10} \\
&- 2 d n_1 n_5 f n_9 n_{11} - 2 d n_3 f n_5 n_4 n_{10} - 2 d n_3 f n_5 n_4 n_{11} + 4 d n_2 n_3 f n_9 n_{11} \\
&+ 4 d n_2 n_3 f n_9 n_{10} - 2 n_4^2 f^3 n_{10} n_{11} - 2 d^2 n_2^3 f n_{11} - 2 d^2 n_2^3 f n_{10} + 4 d^2 n_2^2 f n_5^2 \\
&- 2 d n_2^2 f n_{11}^2 - 2 d n_2^2 f n_{10}^2 + 4 d n_2 f n_5^2 n_{11} - 4 d n_2^2 f n_{10} n_{11} + 4 n_4 f^3 n_5 n_9 n_{10} \\
&+ 4 n_4 f^3 n_5 n_9 n_{11} - 2 d^2 n_1 n_4 f n_5^2 + 2 d^2 n_2^2 n_3 f n_9 + d n_1 n_4 f n_{10}^2 + d n_1 n_4 f n_{11}^2 \\
&- 4 d n_3 f n_9 n_5^2 + 4 d n_2 f n_5^2 n_{10}) th_2^2 + (4 n_5^2 d n_{10} + 4 n_5^2 d n_{11} - 2 n_{10}^2 n_2 d \\
&- 2 n_{11}^2 n_2 d + 4 n_2 d^2 n_5^2 - 2 n_2^2 d^2 n_{10} - 2 n_2^2 d^2 n_{11} - 2 d^2 n_1 n_4 f n_5^2 n_{11} \\
&+ 2 d^2 n_2^2 n_3 f n_9 n_{11} + 2 d^2 n_2^2 n_3 f n_9 n_{10} + d^2 n_2 n_1 n_4 f n_{11}^2 - 4 d^2 n_2 n_3 f n_9 n_5^2 \\
&+ d^2 n_2 n_1 n_4 f n_{10}^2 - 2 d^2 n_1 n_4 f n_5^2 n_{10} - 2 d^2 n_2 n_1 n_5 f n_9 n_{11} - 2 d^2 n_2 n_3 f n_5 n_4 n_{10} \\
&- 2 d^2 n_2 n_3 f n_5 n_4 n_{11} - 2 d^2 n_2 n_1 n_5 f n_9 n_{10} + 2 d^2 n_2 n_1 n_4 f n_{10} n_{11} - 4 n_{10} n_{11} n_2 d \\
&- d^2 n_2^3 f n_{11}^2 - d^2 n_2^3 f n_{10}^2 - 4 d^2 n_2 f n_5^4 + 4 d^2 n_1 n_5^3 f n_9 + 4 d^2 n_3 f n_5^3 n_4 \\
&- 2 d^2 n_2^3 f n_{10} n_{11} + 4 d^2 n_2^2 f n_5^2 n_{10} + 4 d^2 n_2^2 f n_5^2 n_{11}) th_2 - 4 n_5^4 d^2 + 4 n_5^2 d^2 n_{10} n_2 \\
&- n_{10}^2 n_2^2 d^2 + 4 n_5^2 d^2 n_{11} n_2 - 2 n_{10} n_2^2 d^2 n_{11} - n_{11}^2 n_2^2 d^2, -d^2 n_2^2 n_5 n_{10}^2 \\
&- d^2 n_2^2 n_5 n_{11}^2 + 4 d^2 n_2 n_5^3 n_{11} + 4 d^2 n_2 n_5^3 n_{10} - d^2 n_2 n_6 n_5 n_4 n_{11} \\
&- 2 d^2 n_2 n_3 n_5 n_9 n_{11} - d^2 n_2 n_6 n_5 n_4 n_{10} - d^2 n_2 n_7 n_5 n_4 n_{11} - 2 d^2 n_2 n_3 n_5 n_9 n_{10} \\
&+ 2 d^2 n_2 n_3 n_4 n_{10} n_{11} - d^2 n_2 n_7 n_5 n_4 n_{10} - n_5 th_2^4 - 4 d^2 n_5^5 - 2 d^2 n_2^2 n_5 n_{10} n_{11} \\
&+ 2 d^2 n_6 n_5^3 n_4 + 2 d^2 n_7 n_5^3 n_4 + 4 d^2 n_3 n_5^3 n_9 \\
&+ (n_3 d n_4 + n_6 n_9 - 2 n_5 n_{10} + n_4 f n_9 + n_7 n_9 - 2 n_5 n_{11} - 2 n_5 n_2 d) th_2^3 + (2 d n_3 n_4 n_{11} \\
&- d n_4^2 f n_5 + d^2 n_2 n_3 n_4 + n_6 n_9 n_{11} + n_4 f n_9 n_{11} - d n_6 n_5 n_4 + 2 d n_3 n_4 n_{10} - n_5 n_2^2 d^2 \\
&+ n_4 f n_9 n_{10} - n_5 n_{11}^2 - 2 n_5 f n_9^2 + n_6 n_9 n_{10} + d n_2 n_4 f n_9 + n_{10} n_7 n_9 + 2 d n_2 n_6 n_9 \\
&- 2 n_5 n_{10} n_{11} - d n_7 n_5 n_4 + 2 d n_2 n_7 n_9 - 4 d n_5 n_{11} n_2 - n_5 n_{10}^2 - 4 d n_5 n_{10} n_2 \\
&- 2 d n_3 n_5 n_9 + n_{11} n_7 n_9 + 4 d n_5^3) th_2^2 + (d n_2 n_4 f n_9 n_{10} + d n_2 n_4 f n_9 n_{11} + 4 d n_5^3 n_{11} \\
&+ 4 d n_5^3 n_{10} - 2 d^2 n_5 n_2^2 n_{10} - 2 d^2 n_5 n_2^2 n_{11} - 2 d n_2 n_5 n_{11}^2 - 2 d n_2 n_5 n_{10}^2 \\
&+ d^2 n_2^2 n_7 n_9 + d^2 n_2^2 n_6 n_9 - 2 d^2 n_3 n_4 n_5^2 + d n_3 n_4 n_{10}^2 + d n_3 n_4 n_{11}^2 - 2 d n_6 n_9 n_5^2 \\
&- 2 d n_7 n_9 n_5^2 + 2 d^2 n_2 n_3 n_4 n_{10} + 2 d^2 n_2 n_3 n_4 n_{11} - 2 d^2 n_2 n_3 n_5 n_9 - d^2 n_2 n_7 n_5 n_4 \\
&- d^2 n_2 n_6 n_5 n_4 + 2 d n_2 n_7 n_9 n_{10} + 2 d n_2 n_7 n_9 n_{11} - 2 d n_2 n_5 f n_9^2 + 2 d n_2 n_6 n_9 n_{10} \\
&+ 2 d n_2 n_6 n_9 n_{11} - d n_7 n_5 n_4 n_{11} + 2 d n_4 f n_9 n_5^2 - d n_6 n_5 n_4 n_{10} - d n_6 n_5 n_4 n_{11} \\
&- d n_7 n_5 n_4 n_{10} - d n_4^2 f n_5 n_{10} - d n_4^2 f n_5 n_{11} + 2 d n_3 n_4 n_{10} n_{11} - 2 d n_3 n_5 n_9 n_{10} \\
&- 2 d n_3 n_5 n_9 n_{11} - 4 d n_2 n_5 n_{10} n_{11} + 4 d^2 n_5^3 n_2) th_2 - 2 d^2 n_2 n_5^2 n_6 n_9 \\
&- 2 d^2 n_2 n_5^2 n_7 n_9 + d^2 n_2^2 n_{10} n_7 n_9 - 2 d^2 n_3 n_4 n_5^2 n_{10} - 2 d^2 n_3 n_4 n_5^2 n_{11} \\
&+ d^2 n_2^2 n_{11} n_7 n_9 + d^2 n_2^2 n_6 n_9 n_{10} + d^2 n_2^2 n_6 n_9 n_{11} + d^2 n_2 n_3 n_4 n_{11}^2 + d^2 n_2 n_3 n_4 n_{10}^2, \\
&th_2^4 d + (d^2 n_2 - n_8 f + n_{11} d + n_{10} d) th_2^3
\end{aligned}$$

$$\begin{aligned}
& + (d^2 n_{11} n_2 - d n_2 n_8 f - n_8 f n_{10} - n_8 f n_{11} + d^2 n_{10} n_2 + d n_4^2 f - 2 d^2 n_5^2 - 1 + 2 n_9^2 f) th_2^2 \\
& + (-n_{11} - n_2 d + 2 d n_2 n_9^2 f - d n_2 n_8 f n_{10} - d n_2 n_8 f n_{11} - 4 d n_5 n_4 f n_9 + d n_4^2 f n_{11} - n_{10} \\
& + 2 d n_8 f n_5^2 + d n_4^2 f n_{10}) th_2 + 2 n_5^2 d - n_{10} n_2 d - n_{11} n_2 d, (-f n_{10} g^2 - f n_{11} g^2) th_2^5 + (\\
& -4 f n_{11} g^2 n_{10} - 2 d n_2 f n_{10} g^2 + d n_6 n_4 f g^2 + f^3 n_9^2 + d n_7 n_4 f g^2 + f n_{13} g^2 n_9 + f n_{14} g^2 n_9 \\
& - f^2 n_9^2 - 2 f n_{11}^2 g^2 - 2 d n_2 f n_{11} g^2 - 2 f n_{10}^2 g^2 - g^2 - d f^2 n_9^2 + 2 f n_{12} g^2 n_9) th_2^4 + (\\
& d^2 n_2 n_6 n_4 f g^2 + d^2 n_2 n_7 n_4 f g^2 - 3 f n_{11} g^2 n_{10}^2 - 3 f n_{11}^2 g^2 n_{10} - 2 d^2 n_2 f^2 n_9^2 \\
& + 2 d n_2 f^3 n_9^2 - 2 d n_2 f^2 n_9^2 + 2 d^2 f^2 n_9 n_5 n_4 - f n_{10}^3 g^2 - f n_{11}^3 g^2 + 2 f n_{12} g^2 n_9 n_{10} \\
& + 2 f n_{12} g^2 n_9 n_{11} + f n_{13} g^2 n_9 n_{10} + f n_{13} g^2 n_9 n_{11} + f n_{14} g^2 n_9 n_{10} + f n_{14} g^2 n_9 n_{11} \\
& - d^2 n_2^2 f g^2 n_{11} - d^2 n_2^2 f g^2 n_{10} - 2 d n_4 f^3 n_5 n_9 + 2 d f^2 n_9 n_5 n_4 + 4 d f n_{11} g^2 n_5^2 \\
& + 4 d f n_{10} g^2 n_5^2 - 4 d n_2 f n_{11}^2 g^2 - 4 d n_2 f n_{10}^2 g^2 - 2 n_{11} g^2 - 2 n_{10} g^2 \\
& - 8 d n_2 f n_{11} g^2 n_{10} + 2 d n_7 n_4 f g^2 n_{10} + 2 d n_7 n_4 f g^2 n_{11} - 2 d n_7 n_5 f g^2 n_9 \\
& - 2 d f n_{12} g^2 n_5 n_4 - d f n_{13} g^2 n_5 n_4 - 2 n_2 d g^2 + 2 d n_2 f n_{14} g^2 n_9 + 4 d n_2 f n_{12} g^2 n_9 \\
& + 2 d n_2 f n_{13} g^2 n_9 - d f n_{14} g^2 n_5 n_4 + 2 d n_6 n_4 f g^2 n_{10} + 2 d n_6 n_4 f g^2 n_{11} \\
& - 2 d n_6 n_5 f g^2 n_9) th_2^3 + (2 d^2 n_2^2 f n_{12} g^2 n_9 + d^2 n_2^2 f n_{14} g^2 n_9 - 4 d^2 n_2^2 f n_{11} g^2 n_{10} \\
& - 2 d^2 n_2 f^3 n_9 n_5 n_4 + 4 d^2 n_2 f n_{11} g^2 n_5^2 + 4 d^2 n_2 f n_{10} g^2 n_5^2 - d^3 f^2 n_9^2 n_2^2 - d^3 f^2 n_5^2 n_4^2 \\
& + d^2 n_2^2 f^3 n_9^2 - d^2 f^2 n_5^2 n_4^2 - d^2 f^2 n_9^2 n_2^2 + d^2 f^3 n_5^2 n_4^2 - 4 d n_2 n_{10} g^2 - 4 d n_2 n_{11} g^2 \\
& - d^2 n_2^2 g^2 + 4 d g^2 n_5^2 - 2 d^2 n_2^2 f n_{10}^2 g^2 - 2 d^2 n_2^2 f n_{11}^2 g^2 + 4 d f n_{11}^2 g^2 n_5^2 \\
& + 4 d f n_{10}^2 g^2 n_5^2 - 2 d n_2 f n_{11}^3 g^2 - 2 d n_2 f n_{10}^3 g^2 - 2 n_{11} g^2 n_{10} + 2 d^3 f^2 n_9 n_5 n_4 n_2 \\
& + d^2 n_2^2 f n_{13} g^2 n_9 + 2 d^2 f^2 n_9 n_5 n_4 n_2 - 2 d^2 n_7 n_4 f g^2 n_5^2 - 2 d^2 n_6 n_4 f g^2 n_5^2 \\
& - 6 d n_2 f n_{11}^2 g^2 n_{10} + 8 d f n_{11} g^2 n_5^2 n_{10} + d n_7 n_4 f g^2 n_{10}^2 + d n_7 n_4 f g^2 n_{11}^2 \\
& - 4 d f n_{12} g^2 n_9 n_5^2 - 2 d f n_{13} g^2 n_9 n_5^2 - 2 d f n_{14} g^2 n_9 n_5^2 - 2 d^2 n_2 n_6 n_5 f g^2 n_9 \\
& + 2 d^2 n_2 n_7 n_4 f g^2 n_{10} + 2 d^2 n_2 n_7 n_4 f g^2 n_{11} - 2 d^2 n_2 n_7 n_5 f g^2 n_9 - 2 d^2 n_2 f n_{12} g^2 n_5 n_4 \\
& - d^2 n_2 f n_{13} g^2 n_5 n_4 - d^2 n_2 f n_{14} g^2 n_5 n_4 + 2 d^2 n_2 n_6 n_4 f g^2 n_{10} + 2 d^2 n_2 n_6 n_4 f g^2 n_{11} \\
& + 2 d n_6 n_4 f g^2 n_{10} n_{11} - 2 d n_6 n_5 f g^2 n_9 n_{10} - 2 d n_6 n_5 f g^2 n_9 n_{11} - 2 d n_7 n_5 f g^2 n_9 n_{10} \\
& - 2 d n_7 n_5 f g^2 n_9 n_{11} + 2 d n_7 n_4 f g^2 n_{10} n_{11} - 2 d f n_{12} g^2 n_5 n_4 n_{10} - 2 d f n_{12} g^2 n_5 n_4 n_{11} \\
& - d f n_{13} g^2 n_5 n_4 n_{10} - d f n_{13} g^2 n_5 n_4 n_{11} - d f n_{14} g^2 n_5 n_4 n_{10} - d f n_{14} g^2 n_5 n_4 n_{11} \\
& + 2 d n_2 f n_{14} g^2 n_9 n_{10} + 2 d n_2 f n_{14} g^2 n_9 n_{11} + 4 d n_2 f n_{12} g^2 n_9 n_{10} \\
& + 4 d n_2 f n_{12} g^2 n_9 n_{11} + 2 d n_2 f n_{13} g^2 n_9 n_{10} + 2 d n_2 f n_{13} g^2 n_9 n_{11} - n_{11}^2 g^2 - n_{10}^2 g^2 \\
& - 6 d n_2 f n_{11} g^2 n_{10}^2 + d n_6 n_4 f g^2 n_{10}^2 + d n_6 n_4 f g^2 n_{11}^2) th_2^2 + (-3 d^2 n_2^2 f n_{11} g^2 n_{10}^2 \\
& - 3 d^2 n_2^2 f n_{11}^2 g^2 n_{10} + 4 d^2 n_2 f n_{11}^2 g^2 n_5^2 + 4 d^2 n_2 f n_{10}^2 g^2 n_5^2 - 2 d^2 n_2^2 g^2 n_{10} \\
& + 4 d^2 n_2 g^2 n_5^2 + 4 d g^2 n_5^2 n_{10} + 4 d g^2 n_5^2 n_{11} - 2 d n_2 n_{10}^2 g^2 - 2 d n_2 n_{11}^2 g^2 \\
& - 2 d^2 n_2^2 g^2 n_{11} + 2 d^2 n_2 n_6 n_4 f g^2 n_{10} n_{11} - 2 d^2 n_2 n_6 n_5 f g^2 n_9 n_{10}
\end{aligned}$$

$$\begin{aligned}
& -2 d^2 n_2 n_6 n_5 f g^2 n_9 n_{11} - d^2 n_2^2 f n_{10}^3 g^2 - 4 d^2 f n_{11} g^2 n_5^4 - 4 d^2 f n_{10} g^2 n_5^4 \\
& - d^2 n_2^2 f n_{11}^3 g^2 - 4 d n_2 n_{11} g^2 n_{10} + 4 d^2 n_6 n_5^3 f g^2 n_9 + 4 d^2 n_7 n_5^3 f g^2 n_9 \\
& + 4 d^2 f n_{12} g^2 n_5^3 n_4 + 2 d^2 f n_{13} g^2 n_5^3 n_4 + 2 d^2 f n_{14} g^2 n_5^3 n_4 - 2 d^2 n_6 n_4 f g^2 n_5^2 n_{10} \\
& - 2 d^2 n_6 n_4 f g^2 n_5^2 n_{11} + 2 d^2 n_2^2 f n_{12} g^2 n_9 n_{10} + 2 d^2 n_2^2 f n_{12} g^2 n_9 n_{11} \\
& + d^2 n_2^2 f n_{13} g^2 n_9 n_{10} + d^2 n_2^2 f n_{13} g^2 n_9 n_{11} + d^2 n_2^2 f n_{14} g^2 n_9 n_{10} \\
& + d^2 n_2^2 f n_{14} g^2 n_9 n_{11} + 8 d^2 n_2 f n_{11} g^2 n_5^2 n_{10} + d^2 n_2 n_7 n_4 f g^2 n_{10}^2 \\
& + d^2 n_2 n_7 n_4 f g^2 n_{11}^2 - 4 d^2 n_2 f n_{12} g^2 n_9 n_5^2 - 2 d^2 n_2 f n_{13} g^2 n_9 n_5^2 \\
& - 2 d^2 n_2 f n_{14} g^2 n_9 n_5^2 + d^2 n_2 n_6 n_4 f g^2 n_{11}^2 + d^2 n_2 n_6 n_4 f g^2 n_{10}^2 \\
& + 2 d^2 n_2 n_7 n_4 f g^2 n_{10} n_{11} - 2 d^2 n_2 n_7 n_5 f g^2 n_9 n_{10} - 2 d^2 n_2 n_7 n_5 f g^2 n_9 n_{11} \\
& - 2 d^2 n_2 f n_{12} g^2 n_5 n_4 n_{10} - 2 d^2 n_2 f n_{12} g^2 n_5 n_4 n_{11} - d^2 n_2 f n_{13} g^2 n_5 n_4 n_{10} \\
& - d^2 n_2 f n_{13} g^2 n_5 n_4 n_{11} - d^2 n_2 f n_{14} g^2 n_5 n_4 n_{10} - d^2 n_2 f n_{14} g^2 n_5 n_4 n_{11} \\
& - 2 d^2 n_7 n_4 f g^2 n_5^2 n_{11} - 2 d^2 n_7 n_4 f g^2 n_5^2 n_{10} \text{th2} - d^2 n_2^2 n_{10}^2 g^2 - 2 d^2 n_2^2 n_{11} g^2 n_{10} \\
& + 4 d^2 n_2 g^2 n_5^2 n_{10} + 4 d^2 n_2 g^2 n_5^2 n_{11} - 4 d^2 n_5^4 g^2 - d^2 n_2^2 n_{11}^2 g^2]
\end{aligned}$$

> degree(Thetarels3red[2], th2); degree(Thetarels3red[3], th2);
coeff(Thetarels3red[2], th2, 4); coeff(Thetarels3red[3], th2, 4);

4

4

-n5

d

[We obtain a degree 3 polynomial by canceling the lead terms of the two degree 4 polynomials above.
> b3 := Reduce(collect(subs(rules1, simplify(coeff(Thetarels3red[2], th2, 4) * Thetarels3red[3] - coeff(Thetarels3red[3], th2, 4) * Thetarels3red[2])), th2), SandNbasis, tdeg(g, f, d, u, v, t));

$$\begin{aligned}
b3 := & -4 th_2^2 d^2 t g + th_2 d^3 v t - th_2 d^3 u t + th_2^2 d^3 g - 4 th_2^3 t^2 f g + 4 th_2^2 f^2 v t \\
& + (4 th_2 - 4 th_2^2) g d t^2 + (6 th_2^2 - 4 th_2^3) g f t + (2 th_2^2 + 2 th_2^3 + 6 th_2) g d t \\
& + (-th_2^2 + th_2 - th_2^3) d^2 v t + (th_2^3 + 3 th_2 - 3 th_2^2) d^2 u t + (2 th_2^2 - 4 th_2 - 2 th_2^3) d v t \\
& + (-2 th_2^3 + 2 th_2^2) d u t + (-4 th_2^3 + 4 th_2^2) f v t + (th_2 + 4 th_2^2 - 3 th_2^3) g + (th_2^2 - th_2^3) v \\
& + (th_2^2 - th_2^3) u + (2 th_2^3 - 2 th_2^2 + 3 th_2) d g + (th_2^2 - th_2 + th_2^3) f g \\
& + (-2 th_2 + 6 th_2^2 - 8 th_2^3) t g + (th_2 - th_2^2 + th_2^3) f^2 u + (th_2 + 3 th_2^2 + th_2^3) f^2 v \\
& + (-4 th_2^3 - 4 th_2 + 4 th_2^2) g t^2 + (-2 th_2 + 2 th_2^2 - 2 th_2^3) d^2 v + (1 + th_2) d^3 v \\
& + (-3 th_2^3 - 1 + th_2 + 3 th_2^2) d u + (3 th_2^2 - 3 th_2 - 3 th_2^3 - 1) d v + (8 th_2^2 - th_2 - 6 th_2^3) f v \\
& + (4 th_2^2 - 2 th_2^3 - th_2) f u + (1 - th_2) d^3 u + (-th_2 - 2 th_2^2) g d^2 - 2 f^2 th_2^2 g
\end{aligned}$$

[We see that b3 is not identically 0, since the constant term is not 0.

> factor(coeff(b3, th2, 0));

$$d(d-1)(1+d)(u+v)$$

[We eliminate the fusion multiplicities from the theta_i relations.

```
> ThetarelsElim:=subs(rules1,Thetarels);
```

$$\begin{aligned}
\text{ThetarelsElim} := & \left[-th1^2 f - 1 \right. \\
& - \left(-\frac{5}{2}v^2 + uv + \frac{3}{2}u^2 + 2t - \frac{1}{2}v^4 + v^3u + tv^2 - u^3v - 2vut + \frac{1}{2}u^4 + u^2t \right) d th1 \\
& - (2t - v^2 + u^2) f th2 - 2 \left(\frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3 \right) g th3, \\
& - th1 th2 - (2t - v^2 + u^2) d th1 - (2t + 1) f th2 - 2(u + v) g th3, th1 th3 g \\
& - \left(\frac{3}{2}v + \frac{1}{2}u - vt + ut + \frac{1}{2}v^3 - \frac{1}{2}v^2u - \frac{1}{2}u^2v + \frac{1}{2}u^3 \right) d th1 - (u + v) f th2 - \left(\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right) g th3 \\
& - \left(-\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right) g th3, -th3 th2 g - (u + v) d th1 - (vt + 2v - ut) f th2 - t g th3 - (1 + t) g th3, \\
& th2^2 d - 1 - (2t + 1) d th1 - (tv^2 + 2v^2 - 2uv - 2vut + 1 + 2t + u^2t) f th2 \\
& - 2(vt + 2v - ut) g th3, th3^2 f - th3^2 d - th3^2 - 1 - d th1 \left(\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right) \\
& \left. - d th1 \left(-\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2} \right) - f th2 (1 + t) - th2 tf - 3 g th3 v - g th3 u \right]
\end{aligned}$$

```
> with(numtheory):
```

We next proceed to eliminate all but the desired possibility: $t=0$. For each case, we assume $th2$ satisfies a cyclotomic polynomial. If we trust maple to factor the polynomials irreducibly, less work is needed since only linear factors matter. We prefer to have independent confirmation that there are no integer solutions.

If either $th1$ or $th2$ is -1 , get no solutions.

```
> factor(Basis('union'(convert(SandNrels,set),convert(ThetarelsElim,
set),{(th2+1)}),plex(d,f,g,th1,th2,th3,u,v,t)))[1]);
```

$$(2t+1)(6t+1)(4t^4+8t^3+5t^2+t-1)$$

If $th2$ is 1 then get no solutions. (check modulo 2).

```
factor(Basis('union'(convert(SandNrels,set),convert(ThetarelsElim,se
t),{(th2-1)}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];
```

$$\begin{aligned}
(2t+1)(6t+1)(4t^6-6t^5+8v^2t^4+9t^4-28t^3v^2-4t^3-36v^2t^2+3t^2+16v^4t^2+32v^4t \\
+ 8tv^2+1+16v^4+8v^2) \\
(64v^4+104v^2+49-128tv^2-140t+32v^2t^2+156t^2-80t^3+16t^4)
\end{aligned}$$

```
> factor('mod'(simplify(%/((2*t+1)*(6*t+1))),4));
```

$$(t^2-t+1)(2t^3+3t^2+t+1)$$

If $th2$ is a 4th root of 1, no solutions.

```
factor(Basis('union'(convert(SandNrels,set),convert(ThetarelsElim,se
t),{th2^2+1}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];
```

$$(2t+1)(4t^2-4t-1)(4t^6+12t^5+7t^4-6t^3-11t^2-6t-1)$$

```
> factor('mod'(% , 2));
```

$$(t^2 + t + 1)(t^2 - t + 1)$$

If th2 is a 3rd root of unity, no solutions

```
> factor(Basis('union'(convert(SandNrels, set), convert(ThetarelsElim,
set), {cyclotomic(3, th2)})), plex(d, th3, f, g, th2, th1, u, v, t))[1];
```

$$-(1 + 7t + 16t^2 + 18t^3 + 9t^4)(4t^2 + 2t + 1 - 8v^2)$$

```
> 'mod'(4*t^2+2*t+1-8*v^2, 4);
```

$$2t + 1$$

th2^6=1, no solutions.

```
> factor(Basis('union'(convert(SandNrels, set), convert(ThetarelsElim,
set), {cyclotomic(6, th2)})), plex(d, th3, f, g, th2, th1, u, v, t))[1];
```

$$(8t^2 - 4t - 1)(128t^8 + 512t^7 + 904t^6 + 920t^5 + 573t^4 + 210t^3 + 36t^2 - t - 1)$$

If th2 is a 7th root of 1 then t=0, which implies u=1, v=0 as expected.

```
> factor(Basis('union'(convert(SandNrels, set), convert(ThetarelsElim,
set), {cyclotomic(7, th2)})), plex(d, th3, f, g, th2, th1, u, v, t))[1];
```

$$t(1+t)(8t^3 + 4t^2 - 4t - 1)(8t^3 - 12t^2 - 8t - 1)(4096t^{18} + 36864t^{17} + 122880t^{16} + 147456t^{15}$$

$$- 224000t^{14} - 1223936t^{13} - 2439296t^{12} - 2877952t^{11} - 1900768t^{10} + 7200t^9 + 1639832t^8$$

$$+ 2181536t^7 + 1753627t^6 + 998977t^5 + 417004t^4 + 126225t^3 + 26441t^2 + 3446t + 211)$$

```
> 'mod'(4096*t^18+36864*t^17+122880*t^16+147456*t^15-224000*t^14-122
3936*t^13-2439296*t^12-2877952*t^11-1900768*t^10+7200*t^9+1639832*
t^8+2181536*t^7+1753627*t^6+998977*t^5+417004*t^4+126225*t^3+26441
*t^2+3446*t+211, 2);
```

$$t^6 + t^5 + t^3 + t^2 + 1$$

If th2 is a 9th root of 1, no integer solutions.

```
> factor(Basis('union'(convert(SandNrels, set), convert(ThetarelsElim,
set), {cyclotomic(9, th2)})), plex(d, th3, f, g, th2, th1, u, v, t))[1];
```

$$(24t^3 - 6t - 1)(8t^3 - 6t - 1)(-9t - 1 + 63742464t^{20} + 1327104t^{23} - 3422979t^{10}$$

$$+ 150947712t^{18} + 113716224t^{19} + 3881448t^8 + 91239696t^{16} + 174t^2 + 13848192t^{15}$$

$$- 48148515t^{12} + 2085462t^7 + 197052t^5 - 63688896t^{13} + 110592t^{24} + 7464960t^{22} + 3752982t^9$$

$$+ 3848t^3 - 22098258t^{11} + 35046t^4 + 761903t^6 - 45467136t^{14} + 26155008t^{21} + 145846656t^{17})$$

```
> factor('mod'(% , 2));
```

$$(t^6 - t^3 + 1)(t^6 + t^4 + t^3 + t + 1)$$

14th roots of unity slightly trickier. We use the rational-root theorem.

```
> factor(Basis('union'(convert(SandNrels, set), convert(ThetarelsElim,
set), {cyclotomic(14, th2)})), plex(d, th3, f, g, th2, th1, u, v, t))[1];
```

$$(8t^3 + 4t^2 - 4t - 1)(64t^6 + 192t^5 - 208t^4 - 64t^3 + 32t^2 + 12t + 1)(232t + 8 - 5270784t^{20}$$

$$+ 49152t^{23} - 17853112t^{10} - 61902848t^{18} - 22430208t^{19} + 6118813t^8 - 198956704t^{16}$$

$$+ 3114t^2 - 250715392t^{15} - 133342745t^{12} + 4196216t^7 + 608816t^5 - 207623648t^{13} + 4096t^{24}$$

$$+ 147456t^{22} + 2134759t^9 + 25751t^3 - 63466934t^{11} + 146809t^4 + 1872713t^6 - 254526880t^{14}$$

$$- 450560t^{21} - 125888256t^{17})$$

```
> qq:=232*t+8-5270784*t^20+49152*t^23-17853112*t^10-61902848*t^18-22
430208*t^19+6118813*t^8-198956704*t^16+3114*t^2-250715392*t^15-133
```

```

342745*t^12+4196216*t^7+608816*t^5-207623648*t^13+4096*t^24+147456
*t^22+2134759*t^9+25751*t^3-63466934*t^11+146809*t^4+1872713*t^6-2
54526880*t^14-450560*t^21-125888256*t^17;

```

```

qq := 232 t + 8 - 5270784 t^20 + 49152 t^23 - 17853112 t^10 - 61902848 t^18 - 22430208 t^19
+ 6118813 t^8 - 198956704 t^16 + 3114 t^2 - 250715392 t^15 - 133342745 t^12 + 4196216 t^7
+ 608816 t^5 - 207623648 t^13 + 4096 t^24 + 147456 t^22 + 2134759 t^9 + 25751 t^3 - 63466934 t^11
+ 146809 t^4 + 1872713 t^6 - 254526880 t^14 - 450560 t^21 - 125888256 t^17

```

```

> ifactor(coeff(qq,t,24));

```

(2)¹²

```

> ifactor(coeff(qq,t,0));

```

(2)³

```

> seq(subs({t=2^i},qq),i=0..3);

```

-1327120136, -77676358513144, -14361851015873186552, 44307665179982302699048904

Finally, if th2 is an 18th root of 1, no solutions.

```

> factor(Basis('union'(convert(SandNrels,set),convert(TheTarelsElim,
set),{cyclotomic(18,th2)}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];

```

```

(24 t^3 - 6 t - 1)(64 t^6 - 192 t^4 - 32 t^3 + 36 t^2 + 12 t + 1)(45 t + 1 + 313344 t^20 + 49152 t^23
+ 50644665 t^10 - 7218688 t^18 - 2070528 t^19 + 14180562 t^8 - 8066160 t^16 + 870 t^2 + 9663616 t^15
+ 76832805 t^12 + 5467878 t^7 + 406122 t^5 + 64982928 t^13 + 4096 t^24 + 239616 t^22 + 29733882 t^9
+ 9826 t^3 + 69839694 t^11 + 74292 t^4 + 1684269 t^6 + 38219568 t^14 + 563200 t^21 - 11736576 t^17)

```

```

> 'mod'(% , 2);

```

$t + 1 + t^{10} + t^{12} + t^6$

```

>

```