

[ This file contains the computations relevant to the paper "On the classification of non-self-dual modular categories" by Hong and Rowell. This is the case where  $b=0$ ,  $y=-f$  and  $a=g$ .

[ > with(LinearAlgebra): with(Groebner):

[ Using the Galois argument assuming that (01)(34), the orthogonality relations imply  $z=(d-1)$  and  $h1=-1/2(d+1)$ .

> S1:=subs({x=1,y=-f,a=g,b=0},Matrix(5,5,[[1,d,f,g,g],[d,x,y,a,a],[f,y,z,b,b],[g,a,b,h1+I\*h2,h1-I\*h2],[g,a,b,h1-I\*h2,h1+I\*h2]]));

$$S1 := \begin{bmatrix} 1 & d & f & g & g \\ d & 1 & -f & g & g \\ f & -f & z & 0 & 0 \\ g & g & 0 & h1+h2I & h1-h2I \\ g & g & 0 & h1-h2I & h1+h2I \end{bmatrix}$$

> C:=Matrix(5,5,[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,0,1],[0,0,0,1,0]]);

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

[ These relations describe the condition that  $S^2$  is proportional to the "charge conjugation matrix."

> Srels:=factor(convert(evalm(S1^2-K^2\*C),set));

Srels := {0,  $f(-z-1+d)$ ,  $(1+2h1+d)g$ ,  $-f(-z-1+d)$ ,  $2d-f^2+2g^2$ ,  $2f^2+z^2-K^2$ ,

$2g^2+2h1^2-2h2^2$ ,  $2g^2+2h1^2+2h2^2-K^2$ ,  $1+d^2+f^2+2g^2-K^2$ }

> factor(Basis('union'(Srels,{-z-1+d,1+2\*h1+d}),lexdeg([K,d,f,g,z],[h2,h1])));

$[2+z+2h1, 1+2h1+d, g^2+h1^2-h2^2, 2+4h1-2h2^2+2h1^2+f^2, (K-2h2)(K+2h2)]$

> slrules:=solve({2+z+2\*h1, 1+2\*h1+d},{z,h1});

$$slrules := \{h1 = -\frac{d}{2} - \frac{1}{2}, z = d - 1\}$$

> S:=subs(slrules,S1);

$$S := \begin{bmatrix} 1 & d & f & g & g \\ d & 1 & -f & g & g \\ f & -f & d-1 & 0 & 0 \\ g & g & 0 & -\frac{d}{2}-\frac{1}{2}+h2 I & -\frac{d}{2}-\frac{1}{2}-h2 I \\ g & g & 0 & -\frac{d}{2}-\frac{1}{2}-h2 I & -\frac{d}{2}-\frac{1}{2}+h2 I \end{bmatrix}$$

These relations come down to a single relation among the d,f and g (the other allow elimination of K, h1 and h2 and z).

> orthrel := {-2\*d+f^2-2\*g^2};

$$\text{orthrel} := \{-2d + f^2 - 2g^2\}$$

Observe that if d is an integer, then d=1 since (01) interchanges d and 1/d. Thus f=-f/d=-f since f and -f/d is interchanged by (01), which implies f=0. So d is not an integer (for this case).

We use the various symmetries of the  $N_{\{i,j\}^k}$  to write down the fusion matrices in terms of just 14 variables. Note that M4 is just the transpose of M3.

> M1 := Matrix([[0,1,0,0,0],[1,n1,n2,n3,n3],[0,n2,n4,n5,n5],[0,n3,n5,n6,n7],[0,n3,n5,n7,n6]]);

$$M1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & n1 & n2 & n3 & n3 \\ 0 & n2 & n4 & n5 & n5 \\ 0 & n3 & n5 & n6 & n7 \\ 0 & n3 & n5 & n7 & n6 \end{bmatrix}$$

> M2 := Matrix([[0,0,1,0,0],[0,n2,n4,n5,n5],[1,n4,n8,n9,n9],[0,n5,n9,n10,n11],[0,n5,n9,n11,n10]]);

$$M2 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & n2 & n4 & n5 & n5 \\ 1 & n4 & n8 & n9 & n9 \\ 0 & n5 & n9 & n10 & n11 \\ 0 & n5 & n9 & n11 & n10 \end{bmatrix}$$

> M3 := Matrix([[0,0,0,0,1],[0,n3,n5,n7,n6],[0,n5,n9,n11,n10],[1,n6,n10,n12,n13],[0,n7,n11,n14,n12]]);

$$M3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & n3 & n5 & n7 & n6 \\ 0 & n5 & n9 & n11 & n10 \\ 1 & n6 & n10 & n12 & n13 \\ 0 & n7 & n11 & n14 & n12 \end{bmatrix}$$

The matrices must commute, giving a set of diophantine equations.

```
> comrels := `minus` ( `union` ( convert ( evalm ( M1 & * M2 - M2 & * M1 ), set ), convert ( evalm ( M1 & * M3 - M3 & * M1 ), set ), convert ( evalm ( M2 & * M3 - M3 & * M2 ), set ), convert ( evalm ( M3 & * Transpose ( M3 ) - Transpose ( M3 ) & * M3 ), set ) ), { 0 } );
```

```
comrels := { n12 - n13, n13 - n12, n3 n6 + n5 n10 + n7 n13 - n3 n7 - n5 n11 - n7 n14,
  n3 n7 + n5 n11 + n7 n14 - n3 n6 - n5 n10 - n7 n13,
  n6 n5 + n9 n10 + n11 n13 - n7 n5 - n9 n11 - n11 n14,
  n7 n5 + n9 n11 + n11 n14 - n6 n5 - n9 n10 - n11 n13, 1 + n6^2 + n10^2 + n13^2 - n7^2 - n11^2 - n14^2,
  1 + n1 n4 + n2 n8 + 2 n3 n9 - n2^2 - n4^2 - 2 n5^2, n2^2 + n4^2 + 2 n5^2 - 1 - n1 n4 - n2 n8 - 2 n3 n9,
  n3^2 + n5^2 + 2 n7 n6 - n1 n7 - n2 n11 - n3 n12 - n3 n14,
  n5^2 + n9^2 + 2 n11 n10 - n4 n7 - n11 n8 - n9 n12 - n9 n14,
  n7^2 + n11^2 + n14^2 - 1 - n6^2 - n10^2 - n13^2,
  n1 n7 + n2 n11 + n3 n12 + n3 n14 - n3^2 - n5^2 - 2 n7 n6,
  n4 n7 + n11 n8 + n9 n12 + n9 n14 - n5^2 - n9^2 - 2 n11 n10,
  n1 n5 + n2 n9 + n3 n10 + n3 n11 - n3 n2 - n5 n4 - n6 n5 - n7 n5,
  n2 n5 + n4 n9 + n5 n10 + n5 n11 - n3 n4 - n5 n8 - n6 n9 - n7 n9,
  n2 n6 + n4 n10 + n5 n13 + n5 n12 - n3 n5 - n5 n9 - n6 n10 - n7 n11,
  n2 n7 + n4 n11 + n5 n12 + n5 n14 - n3 n5 - n5 n9 - n6 n11 - n7 n10,
  n3 n2 + n5 n4 + n6 n5 + n7 n5 - n1 n5 - n2 n9 - n3 n10 - n3 n11,
  n3 n4 + n5 n8 + n6 n9 + n7 n9 - n2 n5 - n4 n9 - n5 n10 - n5 n11,
  n3 n5 + n5 n9 + n6 n10 + n7 n11 - n2 n6 - n4 n10 - n5 n13 - n5 n12,
  n3 n5 + n5 n9 + n6 n11 + n7 n10 - n2 n7 - n4 n11 - n5 n12 - n5 n14,
  n3 n6 + n5 n10 + n6 n13 + n7 n12 - n3 n7 - n5 n11 - n6 n12 - n7 n14,
  n3 n7 + n5 n11 + n6 n12 + n7 n14 - n3 n6 - n5 n10 - n6 n13 - n7 n12,
  n6 n5 + n9 n10 + n10 n13 + n11 n12 - n7 n5 - n9 n11 - n10 n12 - n11 n14,
  n7 n5 + n9 n11 + n10 n12 + n11 n14 - n6 n5 - n9 n10 - n10 n13 - n11 n12,
  1 + n1 n6 + n2 n10 + n3 n13 + n3 n12 - n3^2 - n5^2 - n6^2 - n7^2,
  1 + n4 n6 + n8 n10 + n9 n13 + n9 n12 - n5^2 - n9^2 - n10^2 - n11^2,
  n3^2 + n5^2 + n6^2 + n7^2 - 1 - n1 n6 - n2 n10 - n3 n13 - n3 n12,
```

$$n5^2 + n9^2 + n10^2 + n11^2 - 1 - n4 n6 - n8 n10 - n9 n13 - n9 n12 \}$$

> indets(comrels);

$$\{n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9\}$$

The commutation relations alone leave 6 degrees of freedom.

> HilbertDimension(comrels, tdeg(n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9));

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The Characteristic polynomials of M1, M2 and M3 are computed in two ways. First from the knowledge of the eigenvalues.

> ch1:=collect(expand(product('(X-S[2,i]/S[1,i])', i=1..5)), X);

ch2:=collect(expand(product('(X-S[3,i]/S[1,i])', i=1..5)), X);

ch3:=collect(expand(product('(X-S[4,i]/S[1,i])', i=1..5)), X);

$$ch1 := 1 + X^5 + \left(-\frac{1}{d} - 1 - d\right)X^4 + \left(\frac{1}{d} + d\right)X^3 + \left(\frac{1}{d} + d\right)X^2 + \left(-\frac{1}{d} - 1 - d\right)X$$

$$ch2 := X^5 + \left(\frac{1}{f} + \frac{f}{d} - \frac{d}{f} - f\right)X^4 + \left(-2 + d - \frac{f^2}{d} + \frac{1}{d}\right)X^3 + \left(f - \frac{f}{d}\right)X^2$$

$$ch3 := X^5 + \left(-g + \frac{1}{g} + \frac{d}{g} - \frac{g}{d}\right)X^4 + \left(\frac{g^2}{d} - \frac{1}{d} + \frac{d}{2g^2} - 2 + \frac{h2^2}{g^2} - d + \frac{1}{4g^2} + \frac{d^2}{4g^2}\right)X^3$$

$$+ \left(-\frac{h2^2}{g} - \frac{3}{4g} + g - \frac{d^2}{4g} - \frac{h2^2}{dg} - \frac{1}{4dg} + \frac{g}{d} - \frac{3d}{4g}\right)X^2 + \left(\frac{1}{2} + \frac{1}{4d} + \frac{d}{4} + \frac{h2^2}{d}\right)X$$

>

set m=(d-1)/f which is an integer.

> simplify({subs({m=(d-1)/f}, coeff(ch2, X, 4)+coeff(ch2, X, 2)+m), subs({m=(d-1)/f}, subs({X=m}, ch2))});

$$\{0\}$$

The form of these polynomials imply further relations.

> simplify({coeff(ch1, X, 0)-1, coeff(ch2, X, 0), coeff(ch2, X, 1), coeff(ch3, X, 0), coeff(ch1, X, 4)-coeff(ch1, X, 1)});

$$\{0\}$$

> factor(ch1); factor(ch2); factor(ch3);

$$\frac{(X+1)(X-1)^2(dX-1)(X-d)}{d}$$

d

$$\frac{X^2(X-f)(fX+1-d)(f+dX)}{df}$$

df

$$\frac{X(X-g)(dX-g)(4X^2g^2+4gXd+4gX+2d+1+d^2+4h2^2)}{4dg^2}$$

4dg<sup>2</sup>

> p1:=CharacteristicPolynomial(M1, X);

p2:=CharacteristicPolynomial(M2, X);

p3:=CharacteristicPolynomial(M3, X);

> factor(p1); factor(p2);

$$(X-n6+n7)(-X^3n4-X^3n6-2X^2n5^2+2n5^2-n4n7-n4n6-X^2+X^4+X^2n1n6+X^2n1n4$$

$$\begin{aligned}
& +X^2 n1 n7 + X^2 n4 n7 + 2 X n1 n5^2 + 2 X n3^2 n4 + X n6 n2^2 + X n2^2 n7 - X^3 n1 - 2 X^2 n3^2 \\
& - X^2 n2^2 - X^3 n7 + X n4 + X n6 + X n7 - 4 X n2 n3 n5 - X n6 n1 n4 - X n1 n4 n7 + X^2 n6 n4) \\
(X+n11-n10)(-X^3 n11 - 2 X^2 n5^2 + 2 n5^2 - n2 n11 - n2 n10 - X^2 + X^4 + 2 X n5^2 n8 \\
& + 2 X n2 n9^2 + X n10 n4^2 + X n11 n4^2 + X n10 + X^2 n11 n8 + X^2 n2 n8 + X^2 n2 n10 + X^2 n2 n11 \\
& - 2 X^2 n9^2 - X^3 n10 - X^3 n8 - X^3 n2 - X^2 n4^2 + X n2 + X^2 n10 n8 - 4 X n4 n5 n9 - X n2 n8 n11 \\
& - X n10 n2 n8 + X n11)
\end{aligned}$$

Observe that the only linear terms of p1 are (X-1) and (X+1), while the linear terms of p2 are X and (X-m)

we may use this later, but for now we avoid using m.

```

> solve(X+n11-n10=(X-m)); solve(X+n11-n10=X);
      {X=X, m=-n11+n10, n10=n10, n11=n11}
      {X=X, n10=n11, n11=n11}
> mlinrels:={ (n10-n11)*(n10-n11-m) };
      mlinrels:={(-n11+n10)(-m-n11+n10)}
> linrels:={ (n6-n7)^2-1 };
      linrels:={(-n7+n6)^2-1}

```

The relations implied by the coefficients of the Characteristic polynomials.

```

> chrels:=factor({coeff(p1,X,0)-1,coeff(p2,X,0),coeff(p2,X,1),coeff(p3,X,0),coeff(p1,X,4)-coeff(p1,X,1)});
chrels:={(-n11+n10)(-2 n5^2+n2 n11+n2 n10), n4 n6^2-n4 n7^2-2 n5^2 n6+2 n5^2 n7-1,
n7^2 n9-2 n11 n5 n7+n14 n5^2-n14 n3 n9+n11^2 n3, 4 n10 n4 n5 n9-2 n2 n10+2 n11 n5^2 n8
-4 n5 n4 n9 n11+2 n5^2-2 n5^2 n8 n10-2 n9^2 n2 n10+n11^2 n4^2-n10^2 n4^2-n11^2 n2 n8
+2 n9^2 n11 n2+n11^2+n10^2 n2 n8-n10^2, -n4-n1-2 n6+n6^2-n7^2+n6^2 n2^2
-4 n2 n3 n5 n6-n1 n4 n6^2-2 n1 n5^2 n7+4 n2 n3 n5 n7+n1 n4 n7^2+2 n1 n5^2 n6
-2 n3^2 n7 n4+2 n3^2 n4 n6+2 n4 n6-n7^2 n2^2-2 n5^2}
> nops(chrels);

```

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these involve the integer m=z/f=(d-1)/f.

```

> intchar:=Vector([1,-1,m,0,0]);
> intcharrels:='union'(convert(evalm(M1*intchar+intchar),set),convert(evalm(M2*intchar-m*intchar),set),convert(evalm(M3*intchar),set),convert(evalm(Transpose(M3)*intchar),set));
intcharrels:={0,-n1+n2 m,-n3+n5 m,-n5+n9 m,-n7+n11 m,1-n6+n10 m,
-n2+n4 m+m,1-n4+n8 m-m^2}
> mrels:='union'({coeff(p2,X,4)+coeff(p2,X,2)+m,subs({X=m},p2)},mlinrels,intcharrels);
mrels:={0,(-n11+n10)(-m-n11+n10),-n1+n2 m,-n3+n5 m,-n5+n9 m,-n7+n11 m,
1-n6+n10 m,-n2+n4 m+m,1-n4+n8 m-m^2,m^5-(2 n10+n8+n2)m^4
-(2 n5^2+2 n9^2+n11^2-n10^2-2 n8 n10-2 n2 n10+1+n4^2-n2 n8)m^3-(-n11^2 n8-n2

```

$$\begin{aligned}
& -2 n10 - 2 n5^2 n8 + 2 n11 n5^2 + 4 n4 n5 n9 - 2 n2 n9^2 - 2 n10 n4^2 + 2 n9^2 n11 - 2 n10 n5^2 \\
& - 2 n10 n9^2 - n11^2 n2 + n8 n10^2 + n2 n10^2 + 2 n10 n2 n8) m^2 - (-n11^2 - 2 n5^2 + 2 n2 n10 + n10^2 \\
& - 2 n11 n5^2 n8 + 4 n5 n4 n9 n11 - 4 n10 n4 n5 n9 - n11^2 n4^2 + n10^2 n4^2 - 2 n9^2 n11 n2 \\
& + 2 n5^2 n8 n10 + 2 n9^2 n2 n10 + n11^2 n2 n8 - n10^2 n2 n8) m + 2 n11 n5^2 - 2 n10 n5^2 - n11^2 n2 \\
& + n2 n10^2, -n8 + n11^2 n8 + 2 n5^2 n8 - 2 n11 n5^2 - 4 n4 n5 n9 + 2 n2 n9^2 + 2 n10 n4^2 - 2 n9^2 n11 \\
& + 2 n10 n5^2 + 2 n10 n9^2 + n11^2 n2 - n8 n10^2 - n2 n10^2 - 2 n10 n2 n8 + m \}
\end{aligned}$$

We take all of the relations together and process them. .

```
> allNrels:=subs({n8=u,n11=v,n13=t,n14=w},`union`(linrels,comrels,ch
rels,mrels)):
```

```
>
```

```
> factor(Basis(allNrels,lexdeg([w, v, t, u,n1,n6, n9,n10,n12, n2,
n3, n4, n5,n7],[m]))):
```

$$\begin{aligned}
& [(2+m^2)(-1-2n7+n4), n3-n5m, n2-n4m-m, 1-n4+2n10m, -n5+n9m, -n4-1+2n6, \\
& -2+n1+2n4-4n7-2m^2-2n7m^2, 1-n4+um-m^2, t-n12, -n7+vm, \\
& (2+m^2)(m^2n5+mn12+wm+2n7mn12-n5-n7n5), \\
& 2n9-2n12-2w-2n12n7m^2+2n7n9-n5m-4n7n12-m^3n5+n7n5m-wm^2-m^2n12, \\
& -n7(m+2n10-u), -n7(-2n7m+4n10+n4m-m-4v), -(2+m^2)(n5m+n7n12-n7w), \\
& -1+n4+2n5^2+2n7^2-3n7-3n4n7, -n5-2n7mn12+n5n4-2n7n5-2m^2n5+2n7wm \\
& -4n10-8v+2u+3m-8n7n10+3n4m+n7m+n7mn4-2n7^2m+8n5n12, 2n5n10 \\
& +2n9-2n12-2w-2n12n7m^2-3n5m-6n7n12-m^3n5+n7n5m-wm^2-m^2n12 \\
& +2n7w, -n4m-m-4n10-8v+2u+4n5n9-8n7n10+n7mn4+n7m-2n7^2m, 2n9 \\
& -2n12-2w-2n12n7m^2-4n5m-6n7n12-m^3n5+n7n5m-wm^2-m^2n12+2n7w \\
& +n5u, \\
& 2n9-2n12-2w-2n12n7m^2-n5m-4n7n12-m^3n5+n7n5m-wm^2-m^2n12+2n5v, \\
& -n4m-9m-4n10-8v+2u-8n7n10-3n7mn4-3n7m+8n5w+6n7^2m, \\
& (n4+3-2n7)(-1-2n7+n4), (-1-2n7+n4)(-2m+n7m+2n10), n9+n4n9-2n12-2w \\
& -2n12n7m^2-3n5m-6n7n12-m^3n5+n7n5m-wm^2-m^2n12+2n7w, \\
& -3n4m+3m-u-4n7n10+n7mn4+3n7m+un4-2n7^2m, -v-2n7n10+n4v, \\
& -(-1-2n7+n4)(n12-w), 1-n4-4n10^2+2un10, \\
& -2+2n4-3n7+2n7^2-4n10^2-n4n7+4vn10, -1+n9^2-n4-2n9n12, n5-2n9n10 \\
& +2n10n12+3mn12-2wn10+un9-2n12u+m^2n5+3n7mn12+4vn12-mn12n4 \\
& -n7wm, -4n9n10+2n10n12-3m^2n5-5n7mn12-mn12+4n9v+2wn10+3n7wm \\
& -2n5-4vn12+mn12n4, -3-n4-n7+2n7^2-n4n7+2n9w-2n9n12, \\
& 2-2n4-4n10^2-m^2+u^2, -2+2uv+2n4-5n7+2n7^2-4n10^2-n4n7, 2n5+m^2n5 \\
& +3n7mn12+2mn12+4vn12-4wn10-wm-mn12n4-n7wm-n12u+uw,
\end{aligned}$$

$$\begin{aligned}
& -1 + n4 + 2 v^2 + 2 n7^2 - 2 n10^2 - n7 - n4 n7, \\
& -2 n5 - 2 n10 n12 - m^2 n5 + n7 m n12 + 4 v w - 2 w n10 + m n12 - m n12 n4 + n7 w m, \\
& -3 - n4 - n7 + 2 n7^2 - n4 n7 + 2 w^2 - 2 n12^2, 3 n5 + 2 n7 n5 - n5 n7^2 + 2 n12 n7^2 m + m^2 n5 \\
& + 3 n7 m n12 - m n12 + n7 m^2 n5 - 3 w m - 2 m n12 n4 - 3 n7 w m, -n12 - 2 n12 n7 m^2 - 2 n5 m \\
& - 6 n7 n12 - m^3 n5 - w m^2 - m^2 n12 + n12 n4 + 3 n7 w + w n7^2 - 3 n12 n7^2 + n7 n12 n4, \\
& 2 - 2 n4 + 3 n7 + 4 n12 n7 w + 3 n7^2 - 3 n4 n7 + 2 n7^3 - n4 n7^2 - 4 n12^2 n7 + 4 m^2 + 4 n7 m^2, 1 \\
& - n4 - 13 n7 - 4 m^2 n12 w - 18 n7^2 + 3 n4 n7 + 8 n9 n12 + 8 n10^2 - 8 n12^2 - 4 m^2 n12^2 + 4 m^4 n7 \\
& - 4 n7^3 - 8 n7^2 m^2 + 4 n4 n7^2 + 8 n7 n10^2 + 4 m^2 - 8 n12^2 n7 m^2 + 4 m^4 - 4 n7 m^2 - 16 n12^2 n7 \\
& - 8 w n12, 3 n5 - 2 n7 n5 - n5 n7^2 + m^2 n5 + 3 n7 m n12 - 4 n10 n12 n7 - m n12 + 4 w n10 n7 \\
& - n7 m^2 n5 - 3 w m - 2 m n12 n4 - 3 n7 w m + 2 n7^2 w m, -2 n7^3 m + 20 n10 - 24 v - 2 u + 4 m \\
& - 8 n7 n10 + 8 n10 n12^2 - 8 n7 m n12^2 + 4 m n12^2 n4 + 6 n4 m - 5 n7 m + 8 w n10 n12 + n7 m n4 \\
& + m n4 n7^2 - 16 n12^2 v - 4 m n12^2 - 3 n7^2 m, -n5 m - m^3 n5 - 2 n12 - 2 w - m^2 n12 + 2 n9 \\
& - w m^2 - 4 n7 n12 - 2 n12 n7 m^2 - n12 n7^2 + w n7^2 - 2 n10^2 n12 + 2 w n10^2]
\end{aligned}$$

$m > 0$  an integer implies:

```

> factor(Basis('union'(allNrels, {-1-2*n7+n4, m^2*n5+m*n12+w*m+2*n7*m*
n12-n5-n7*n5, n5*m+n7*n12-n7*w}), plex(n1, n6, n9, n12, n2, n3, n4, w,
v, n10, t, u, n5, n7, m))) ;

```

$$\begin{aligned}
& [n5^2 - 2 n7^2 - 2 n7, -2 n7 - m^2 + u m, (1 + n7)(4 n7 t + 3 n5 m - n5 u), \\
& -n7 - n7^2 + n5 m t + m^2 + n7 m^2, n7 (3 m - u + 2 n5 t - u n7 + 3 n7 m), \\
& 6 n7 + 3 m^2 - u^2 - u^2 n7 + 2 u t n5 + 6 n7^2 + 3 n7 m^2, -n7 + n10 m, n7 (m + 2 n10 - u), \\
& n5 (m + 2 n10 - u), 2 n7 + m^2 + 2 u n10 - u^2, 4 n7 + 4 n10^2 + m^2 - u^2, \\
& -6 n7 m - 3 m + 2 n10 + 4 v - u - 4 n5 t + 2 u n7, -n5 + m^2 n5 - n7 n5 + 2 n7 m t + w m + t m, \\
& -n5 m - n7 t + n7 w, -2 n7 m - 2 m - n5 t + n5 w, \\
& -3 n5 m - 2 n5 u + 2 t m^2 - n7 n5 m - u^2 t + m^3 n5 + 2 n7 m^2 t + u^2 w, -8 m - 8 u + 32 n10 \\
& - 19 n7 m + 7 u n7 - 22 n5 t - 2 m^3 + 5 n7^2 m - 2 t^2 m - u n7^2 - 4 u t^2 + 12 n10 t^2 - 2 n7 m^3 \\
& - 8 n7 m t^2 - 8 t^3 n5 + 4 u n7 t^2 + 2 t w u, \\
& -n5 - 5 t m + n7 n5 + 2 n10 t - m^2 n5 - 8 n7 m t - 4 t^2 n5 + 2 u t n7 - u w + 4 w n10, \\
& -2 n7 - 2 - t^2 + w^2, -1 - 2 n7 + n4, n3 - n5 m, n2 - 2 n7 m - 2 m, n12 - t, \\
& -3 n5 m + 2 n9 - 2 t - 2 w - 4 n7 t + n5 u, n6 - n7 - 1, n1 - 2 n7 m^2 - 2 m^2]
\end{aligned}$$

assuming  $n7=0$  gives:

```

> factor(Basis('union'(allNrels, {-1-2*n7+n4, m^2*n5+m*n12+w*m+2*n7*m*
n12-n5-n7*n5, n5*m+n7*n12-n7*w, n7}), plex(n1, n6, n9, n10, n12, n2, n3,
n4, w, v, t, u, n5, n7, m))) ;

```

$$\begin{aligned}
& [m^2, n7, n5 m, n5^2, u m, n5 u, u^2, v m, n5 v, u v, v^2, w m + t m - n5, -n5 t - 2 m + n5 w, \\
& -32 v + 4 u + 20 m + 21 n5 t - 12 t^2 v + u t^2 + 8 t^3 n5 + 8 t^2 m + t w u,
\end{aligned}$$

$$4vt - ut - 5n5 - 8t^2n5 - 8tm + 8vw - uw, w^2 - 2 - t^2, n4 - 1, n3, n2 - 2m, n12 - t, \\ -4n5t - u + 4v + 2n10 - 3m, n9 - w - t, n6 - 1, n1]$$

Thus we may assume  $n7$  not zero, since then we have  $m=0$  and  $d=1$ . so we see that

$$-u+2*n10+m=0, m+2*v-u=0$$

```
> factor(Basis('union'(allNrels, {-1-2*n7+n4, m^2*n5+m*n12+w*m+2*n7*m*
n12-n5-n7*n5, n5*m+n7*n12-n7*w, -u+2*n10+m, m+2*v-u}), plex(n1, n6,
n9, n10, n12, n2, n3, n4, w, v, t, u, n5, n7, m)));
```

$$[n5^2 - 2n7^2 - 2n7, -2n7 - m^2 + um, (1+n7)(4n7t + 3n5m - n5u),$$

$$3m - u + 2n5t - un7 + 3n7m, m + 2v - u, -n5 + m^2n5 - n7n5 + 2n7mt + wmt + tm, \\ -n5m - n7t + n7w, -m - u - n7m - un7 + 2n5w,$$

$$-3n5 + 2tm - ut - n7n5 + m^2n5 + 2n7mt + uw, -2n7 - 2 - t^2 + w^2, -1 - 2n7 + n4, n3 - n5m, \\ n2 - 2n7m - 2m, n12 - t, m + 2n10 - u, -3n5m + 2n9 - 2t - 2w - 4n7t + n5u, n6 - n7 - 1, \\ n1 - 2n7m^2 - 2m^2]$$

$n7+1$  not 0. also observe that we have a diophantine equation in  $n5$  and  $n7$ , which has  $n5=0$  iff  $n7=0$ . So  $n5$  is not zero.

```
> intrels:=factor(Basis('union'(allNrels, {-1-2*n7+n4, m^2*n5+m*n12+w*
m+2*n7*m*n12-n5-n7*n5, n5*m+n7*n12-n7*w, -u+2*n10+m, m+2*v-u, 4*n7*t+3
*n5*m-n5*u}), lexdeg([m, n1, n6, n9, n10, n12, n2, n3, n4, n5, n7], [w, v,
t, u])));
```

$$intrels := [-ut + vt - uw + 3vw, -2 - t^2 - 2uv + 4v^2 + w^2,$$

$$3u - 9v + ut^2 - 4t^2v + 3u^2v - 15v^2u + 18v^3 + twu, n7 + 2v^2 - uv, 3n5 + 4vt - uw - ut,$$

$$-1 - 2uv + 4v^2 + n4, -u^2t + 16vut - 24tv^2 - u^2w + 9n3,$$

$$-6u - 6u^2v + 4ut^2 + 4twu + 12v^2u - 16t^2v + 9n2, n12 - t, -v + n10, n9 - w - t,$$

$$-1 - uv + 2v^2 + n6, -6u^2 - 6u^3v + 8u^2t^2 + 8wu^2t + 12u^2v^2 - 80ut^2v + 96t^2v^2 + 27n1,$$

$$m + 2v - u]$$

```
> rules1:='union'({n8=u, n11=v, n13=t, n14=w}, solve({seq(intrels[i], i=4
..14)}));
```

$$rules1 := \{m = -2v + u, n1 = \frac{2u^2}{9} + \frac{2u^3v}{9} - \frac{8u^2t^2}{27} - \frac{8wu^2t}{27} - \frac{4u^2v^2}{9} + \frac{80ut^2v}{27} - \frac{32t^2v^2}{9},$$

$$n10 = v, n11 = v, n12 = t, n13 = t, n14 = w, n2 = \frac{2u}{3} + \frac{2u^2v}{3} - \frac{4ut^2}{9} - \frac{4twu}{9} - \frac{4v^2u}{3} + \frac{16t^2v}{9},$$

$$n3 = \frac{u^2t}{9} - \frac{16vut}{9} + \frac{8tv^2}{3} + \frac{u^2w}{9}, n4 = 1 + 2uv - 4v^2, n5 = -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3},$$

$$n6 = 1 + uv - 2v^2, n7 = -2v^2 + uv, n8 = u, n9 = t + w, t = t, u = u, v = v, w = w]$$

```
> indets(subs(rules1, 'union'(allNrels, {-1-2*n7+n4, m^2*n5+m*n12+w*m+2
*n7*m*n12-n5-n7*n5, n5*m+n7*n12-n7*w, -u+2*n10+m, m+2*v-u, 4*n7*t+3*n5
*m-n5*u})));
```

$$\{t, u, v, w\}$$

```
> agb:=factor(Basis(subs(rules1, 'union'(allNrels, {-1-2*n7+n4, m^2*n5+
```

```
m*n12+w*m+2*n7*m*n12-n5-n7*n5,n5*m+n7*n12-n7*w,-u+2*n10+m,m+2*v-u,
4*n7*t+3*n5*m-n5*u})),plex( u, v, t,w )));
```

```
agb :=
```

```
[2 t v^2 - t^3 - 2 w v^2 - t^2 w + t w^2 + w^3 - 2 t - 2 w, u t - v t + u w - 3 v w, 2 + t^2 + 2 u v - 4 v^2 - w^2]
> M1s := subs(rules1, M1); M2s := subs(rules1, M2); M3s := subs(rules1, M3);
```

```
M1s :=
```

```
[0, 1, 0, 0, 0]
```

$$\left[ 1, \frac{2u^2}{9} + \frac{2u^3v}{9} - \frac{8u^2t^2}{27} - \frac{8wu^2t}{27} - \frac{4u^2v^2}{9} + \frac{80ut^2v}{27} - \frac{32t^2v^2}{9}, \right. \\ \left. \frac{2u}{3} + \frac{2u^2v}{3} - \frac{4ut^2}{9} - \frac{4twu}{9} - \frac{4v^2u}{3} + \frac{16t^2v}{9}, \frac{u^2t}{9} - \frac{16vut}{9} + \frac{8tv^2}{3} + \frac{u^2w}{9}, \right. \\ \left. \frac{u^2t}{9} - \frac{16vut}{9} + \frac{8tv^2}{3} + \frac{u^2w}{9} \right]$$

$$\left[ 0, \frac{2u}{3} + \frac{2u^2v}{3} - \frac{4ut^2}{9} - \frac{4twu}{9} - \frac{4v^2u}{3} + \frac{16t^2v}{9}, 1 + 2uv - 4v^2, -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}, \right. \\ \left. -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3} \right]$$

$$\left[ 0, \frac{u^2t}{9} - \frac{16vut}{9} + \frac{8tv^2}{3} + \frac{u^2w}{9}, -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}, 1 + uv - 2v^2, -2v^2 + uv \right] \\ \left[ 0, \frac{u^2t}{9} - \frac{16vut}{9} + \frac{8tv^2}{3} + \frac{u^2w}{9}, -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}, -2v^2 + uv, 1 + uv - 2v^2 \right]$$

```
M2s :=
```

```
[0, 0, 1, 0, 0]
```

$$\left[ 0, \frac{2u}{3} + \frac{2u^2v}{3} - \frac{4ut^2}{9} - \frac{4twu}{9} - \frac{4v^2u}{3} + \frac{16t^2v}{9}, 1 + 2uv - 4v^2, -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}, \right. \\ \left. -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3} \right]$$

```
[1, 1 + 2uv - 4v^2, u, t + w, t + w]
```

$$\left[ 0, -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}, t + w, v, v \right]$$

$$\left[ 0, -\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}, t + w, v, v \right]$$

$$M3s := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{u^2 t}{9} - \frac{16 v u t}{9} + \frac{8 t v^2}{3} + \frac{u^2 w}{9} & -\frac{4 v t}{3} + \frac{u w}{3} + \frac{u t}{3} & -2 v^2 + u v & 1 + u v - 2 v^2 \\ 0 & -\frac{4 v t}{3} + \frac{u w}{3} + \frac{u t}{3} & t + w & v & v \\ 1 & 1 + u v - 2 v^2 & v & t & t \\ 0 & -2 v^2 + u v & v & w & t \end{bmatrix}$$

>

The unique positive character is psi0.

```
> psi0:=Vector([1,d,f,g,g]): psi2:=subs({m=(d-1)/f},intchar):
> psirels:=numer('minus'('union'(convert(evalm(M1s&*psi0-psi0[2]*psi0),set),convert(evalm(M2s&*psi0-psi0[3]*psi0),set),convert(evalm(M3s&*psi0-psi0[4]*psi0),set),convert(evalm(M1s&*psi2-psi2[2]*psi2),set),convert(evalm(M2s&*psi2-psi2[3]*psi2),set),convert(evalm(M3s&*psi2-psi2[4]*psi2),set)),{0})));
```

$psirels := \{v(-uf+2vf+d-1), -2dv^2+duv+vf+gw+tg-g^2,$

$1+d+duv-2dv^2+vf+2tg-g^2, -4dvt+duw+dut+3ft+3fw+6vg-3fg,$

$-utf+4vtf-uwf+3td-3t+3wd-3w, -2uvf^2+4v^2f^2+ufd-uf-d^2+2d-1,$

$1+d+2duv-4dv^2+uf+2tg+2gw-f^2,$

$16vutf-24tv^2f-u^2tf-u^2wf-12dvt+12vt+3duw-3uw+3dut-3ut,$

$du^2t-16dvut+24dvt^2+du^2w-12vtf+3uwf+3utf+9g+18guv-36gv^2-9dg,$

$-6uf-6fu^2v+4fut^2+4ftwu+12fv^2u-16ft^2v+18d-18+18duv-18uv-36dv^2$

$+36v^2, 6du+6du^2v-4dut^2-4dwtu-12dv^2u+16dt^2v+9f+18uvf-36v^2f$

$-24gvt+6guw+6gut-9df, 27+6du^2+6du^3v-8du^2t^2-8dwu^2t-12du^2v^2$

$+80dut^2v-96dt^2v^2+18uf+18fu^2v-12fut^2-12ftwu-36fv^2u+48ft^2v+6gu^2t$

$-96gvut+144gtv^2+6gu^2w-27d^2, -6u^2f-6u^3vf+8u^2t^2f+8wu^2tf+12u^2v^2f$

$-80ut^2vf+96t^2v^2f+18du-18u+18du^2v-18u^2v-12dut^2+12ut^2-12dwtu$

$+12twu-36dv^2u+36v^2u+48dt^2v-48t^2v\}$

We combine the nontrivial relations obtained so far.

```
> SandNrels:='union'(psirels,convert(agb,set),orthrel);
```

$SandNrels := \{v(-uf+2vf+d-1), -2d+f^2-2g^2, ut-vt+uw-3vw,$

$2+t^2+2uv-4v^2-w^2, -2dv^2+duv+vf+gw+tg-g^2,$

$1+d+duv-2dv^2+vf+2tg-g^2, -4dvt+duw+dut+3ft+3fw+6vg-3fg,$

$-utf+4vtf-uwf+3td-3t+3wd-3w, -2uvf^2+4v^2f^2+ufd-uf-d^2+2d-1,$

$1+d+2duv-4dv^2+uf+2tg+2gw-f^2, 2tv^2-t^3-2wv^2-t^2w+tw^2+w^3-2t-2w,$

```

16 v u t f - 24 t v^2 f - u^2 t f - u^2 w f - 12 d v t + 12 v t + 3 d u w - 3 u w + 3 d u t - 3 u t,
d u^2 t - 16 d v u t + 24 d t v^2 + d u^2 w - 12 v t f + 3 u w f + 3 u t f + 9 g + 18 g u v - 36 g v^2 - 9 d g,
-6 u f - 6 f u^2 v + 4 f u t^2 + 4 f t w u + 12 f v^2 u - 16 f t^2 v + 18 d - 18 + 18 d u v - 18 u v - 36 d v^2
+ 36 v^2, 6 d u + 6 d u^2 v - 4 d u t^2 - 4 d t w u - 12 d v^2 u + 16 d t^2 v + 9 f + 18 u v f - 36 v^2 f
- 24 g v t + 6 g u w + 6 g u t - 9 d f, 27 + 6 d u^2 + 6 d u^3 v - 8 d u^2 t^2 - 8 d w u^2 t - 12 d u^2 v^2
+ 80 d u t^2 v - 96 d t^2 v^2 + 18 u f + 18 f u^2 v - 12 f u t^2 - 12 f t w u - 36 f v^2 u + 48 f t^2 v + 6 g u^2 t
- 96 g v u t + 144 g t v^2 + 6 g u^2 w - 27 d^2, -6 u^2 f - 6 u^3 v f + 8 u^2 t^2 f + 8 w u^2 t f + 12 u^2 v^2 f
- 80 u t^2 v f + 96 t^2 v^2 f + 18 d u - 18 u + 18 d u^2 v - 18 u^2 v - 12 d u t^2 + 12 u t^2 - 12 d t w u
+ 12 t w u - 36 d v^2 u + 36 v^2 u + 48 d t^2 v - 48 t^2 v}

```

```
> indets(SandNrels);
```

```
{d, f, g, t, u, v, w}
```

```
> SandNbasis := factor(Basis(SandNrels, plex(d, f, g, u, v, t, w)));
```

```
SandNbasis := [2 t v^2 - t^3 - 2 w v^2 - t^2 w + t w^2 + w^3 - 2 t - 2 w, u t - v t + u w - 3 v w,
```

```
2 + t^2 + 2 u v - 4 v^2 - w^2,
```

```
-2 + t^2 - 4 v^2 + 3 w^2 + 2 g^2 - 2 g w - g w^3 - 6 t g - g t^3 + w g t^2 + g w^2 t + 4 t w,
```

```
2 f - 4 v - w^2 f + t^2 f - 2 g v t + 2 g w v, t^2 - 2 v^2 + w^2 - g w + v f - t g + 2 t w,
```

```
2 + 2 t^2 - 4 v^2 + 2 w^2 + u f - 3 g w - t g + 4 t w,
```

```
-v t + f g - g w^2 v - f t - f w - v g t^2 + 2 g v t w + v w - 2 v g,
```

```
t^2 + 4 t w + 3 w^2 - 4 v^2 - 4 t g - 4 g w - g t^3 + w g t^2 + g w^2 t - g w^3 + f^2, -g w + 1 + t g + d]
```

```
>
```

The next step is to introduce the relations involving the roots of unity theta\_i. The first goal is to show that th2 satisfies a nontrivial degree 3 (or less) polynomial in Q[d].

```
> psith0 := Vector([1, d*th1, f*th2, g*th3, g*th3]):
```

```
> Thetarels := simplify(subs({}, [th1^2*S[2,2] - (sum('M1[k,2]*psith0[k]', k=1..5)), th1*th2*S[2,3] - (sum('M1[k,3]*psith0[k]', k=1..5)), th1*th3*S[2,4] - (sum('M1[k,4]*psith0[k]', k=1..5)), th3*th2*S[4,3] - (sum('M2[k,4]*psith0[k]', k=1..5)), th2^2*S[3,3] - (sum('M2[k,3]*psith0[k]', k=1..5)), th3^2*(S[4,4]+S[4,5]) - (sum(' (M3[k,4]+M3[k,5])*psith0[k]', k=1..5))]);
```

```
Thetarels := [th1^2 - 1 - n1 d th1 - n2 f th2 - 2 n3 g th3, -th1 th2 f - n2 d th1 - n4 f th2 - 2 n5 g th3,
```

```
th1 th3 g - n3 d th1 - n5 f th2 - n6 g th3 - n7 g th3, -n5 d th1 - n9 f th2 - n10 g th3 - n11 g th3,
```

```
th2^2 d - th2^2 - 1 - n4 d th1 - n8 f th2 - 2 n9 g th3, -th3^2 d - th3^2 - 1 - d th1 n6 - d th1 n7
```

```
- f th2 n10 - f th2 n11 - 2 g th3 n12 - g th3 n13 - g th3 n14]
```

```
>
```

we immediately find that th1=1.

```
> factor(Basis('union'(SandNrels, convert(subs(rules1, Thetarels), set)), tdeg(th1, u, v, t, w, d, th2, th3, f, g)))[9];
```

```
g th3 (th1 - 1)
```

```
> Thetarels1 := factor( numer(subs({th1=1}, Thetarels) ) );
```

```

Thetarels1 := [-n1 d - n2 f th2 - 2 n3 g th3, -f th2 - n2 d - n4 f th2 - 2 n5 g th3,
  g th3 - n3 d - n5 f th2 - n6 g th3 - n7 g th3, -n5 d - n9 f th2 - n10 g th3 - n11 g th3,
  th2^2 d - th2^2 - 1 - n4 d - n8 f th2 - 2 n9 g th3,
  -th3^2 d - th3^2 - 1 - d n6 - d n7 - f th2 n10 - f th2 n11 - 2 g th3 n12 - g th3 n13 - g th3 n14]

```

```

> indets(Thetarels1);
      {d, f, g, n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9, th2, th3}

```

```

> map(degree, Thetarels1, th3);
      [1, 1, 1, 1, 1, 2]

```

```

> solve(Thetarels1[2], th3);
      -  $\frac{f th2 + n2 d + n4 f th2}{2 n5 g}$ 

```

We substitute back in to eliminate th3 and take at the numerators of the resulting rational functions, which give us our new relations.

```

> Thetarels2 := factor( numer( subs( { th3 = solve( Thetarels1[2], th3 ) }, Thetarels1 ) ) );

```

```

Thetarels2 := [-n1 d n5 - n5 f th2 n2 + n3 f th2 + n3 n2 d + n3 n4 f th2, 0, -f th2 - n2 d - n4 f th2
  - 2 n3 d n5 - 2 n5^2 f th2 + n6 f th2 + n6 n2 d + n6 n4 f th2 + n7 f th2 + n7 n2 d + n7 n4 f th2,
  -2 n5^2 d - 2 n9 f th2 n5 + f th2 n10 + n10 n2 d + n10 n4 f th2 + f th2 n11 + n11 n2 d + n11 n4 f th2,
  th2^2 d n5 - th2^2 n5 - n5 - n4 d n5 - n8 f th2 n5 + n9 f th2 + n9 n2 d + n9 n4 f th2, -n4^2 f^2 th2^2
  - 2 n4 f^2 th2^2 - 2 d n4 f^2 th2^2 - 2 d^2 n4 f th2 n2 - 2 d^2 f th2 n2 + 4 n12 n5 g^2 n4 f th2
  + 2 n13 n5 g^2 n4 f th2 + 2 n14 n5 g^2 n4 f th2 - 2 f th2 n2 d - n2^2 d^3 - n2^2 d^2 - 4 n5^2 g^2
  - 4 d n7 n5^2 g^2 - 4 d n6 n5^2 g^2 - d n4^2 f^2 th2^2 + 4 n12 n5 g^2 f th2 + 4 n12 n5 g^2 n2 d
  + 2 n13 n5 g^2 f th2 + 2 n13 n5 g^2 n2 d + 2 n14 n5 g^2 f th2 + 2 n14 n5 g^2 n2 d - th2^2 f^2
  - 2 n4 f th2 n2 d - d f^2 th2^2 - 4 f th2 n10 n5^2 g^2 - 4 f th2 n11 n5^2 g^2]

```

```

> Thetarels2red := factor( Reduce( subs( rules1, Thetarels2 ), SandNbasis, plex(d, f, g, u, v, t, w, th2) ) );

```

```

> map(degree, Thetarels2red, th2);

```

```

      [-∞, -∞, -∞, -∞, 2, 2]

```

```

> factor( [coeff(Thetarels2red[5], th2, 2), coeff(Thetarels2red[5], th2, 1),
  coeff(Thetarels2red[5], th2, 0)] );

```

```

[9 v (-w + t) (t g - g w + 2), -9 g w^2 v - 18 v t + 18 f w + 18 f t - 9 v g t^2 + 18 g v t w + 18 v w,
  9 v (-w + t) (t g - g w + 2)]

```

provided v(t-w) is not zero, th2 satisfies a nonzero degree 2 polynomial (observe that if g is rational, so is d which is a contradiction). The following shows that if v(t-w)=0, then d=1.

```

> factor( Basis( 'union'( SandNrels, { v*(t-w) } ), plex(f, g, u, v, t, w, d) ) );

```

```

[(d - 1)(d + 1), w(d - 1), t(d - 1), -w^2 + 1 + t^2 + d, v(d + 1), v w, v t, u(d + 1), u w, u t,
  2 u v + 1 - 4 v^2 - d, d g + g - 2 w - 2 t, (-w + g - t) w, t g + 1 - t w - w^2 + d,
  2 g^2 + 1 - 4 v^2 - 4 t w - 4 w^2 + 3 d, d f - f + 4 v, v(-2 v + f), u f + 1 - 4 v^2 - d, f g - 2 v g - f w - f t,

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[ f^2 + 1 - 4 v^2 - 4 t w - 4 w^2 + d]
[ > with(numtheory):
[ Since [Q[d,f,g]:Q]=2, th2 satisfies a cyclotomic polynomial of degree 1,2 or 4.
[ > invphi(4); invphi(2);
[
[ [5, 8, 10, 12]
[ [3, 4, 6]
[ > ThetaElim:=subs(rules1, 'union'(convert(Thetarels, set), {th1-1}));
ThetaElim := {th1 - 1, -\left(-\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}\right) d th1 - (t+w) f th2 - 2vg th3, -th1 th2 f
- \left(\frac{2u}{3} + \frac{2u^2v}{3} - \frac{4ut^2}{9} - \frac{4twu}{9} - \frac{4v^2u}{3} + \frac{16t^2v}{9}\right) d th1 - (1+2uv-4v^2) f th2
- 2\left(-\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}\right) g th3, th1^2 - 1
- \left(\frac{2u^2}{9} + \frac{2u^3v}{9} - \frac{8u^2t^2}{27} - \frac{8wu^2t}{27} - \frac{4u^2v^2}{9} + \frac{80ut^2v}{27} - \frac{32t^2v^2}{9}\right) d th1
- \left(\frac{2u}{3} + \frac{2u^2v}{3} - \frac{4ut^2}{9} - \frac{4twu}{9} - \frac{4v^2u}{3} + \frac{16t^2v}{9}\right) f th2
- 2\left(\frac{u^2t}{9} - \frac{16vut}{9} + \frac{8tv^2}{3} + \frac{u^2w}{9}\right) g th3, th1 th3 g - \left(\frac{u^2t}{9} - \frac{16vut}{9} + \frac{8tv^2}{3} + \frac{u^2w}{9}\right) d th1
- \left(-\frac{4vt}{3} + \frac{uw}{3} + \frac{ut}{3}\right) f th2 - (1+uv-2v^2) g th3 - (-2v^2+uv) g th3,
th2^2 d - th2^2 - 1 - (1+2uv-4v^2) d th1 - u f th2 - 2(t+w) g th3,
-th3^2 d - th3^2 - 1 - d th1 (-2v^2+uv) - d th1 (1+uv-2v^2) - 2 f th2 v - 3 g th3 t - g th3 w}
[ > indets(ThetaElim);
[
[ {d, f, g, t, th1, th2, th3, u, v, w}
[ > bases1:=i->factor(Basis('union'(SandNrels, ThetaElim, {cyclotomic(i,
th2})), plex(d, f, g, th1, th3, th2, u, v, t, w)))[1];
bases1 := i -> factor(Groebner:-Basis(
'union'(SandNrels, ThetaElim, {numtheory:-cyclotomic(i, th2)}),
plex(d, f, g, th1, th3, th2, u, v, t, w)))_1
[ > map(bases1, invphi(4));
[(22201 - 19008 w^6 + 20736 w^8 - 201456 w^4 - 65532 w^2)(t^2 - 2 t w + 4 + w^2),
(-36481 - 89856 w^6 + 10368 w^8 + 159984 w^4 + 19704 w^2)(t^2 - 2 t w + 4 + w^2),
(-58081 - 302400 w^6 + 103680 w^8 + 187920 w^4 + 75660 w^2)(t^2 - 2 t w + 4 + w^2),
(6 w^2 + 18 w + 13)(6 w^2 - 18 w + 13)(288 w^4 + 24 w^2 - 121)(t^2 - 2 t w + 4 + w^2)]
[ > map(bases1, invphi(2));
[(9 w^2 + 4)(t^2 - 2 t w + 4 + w^2), (-1 + 60 w^2 + 72 w^4)(t^2 - 2 t w + 4 + w^2), t^2 - 2 t w + 4 + w^2]

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clearly have no solutions.
> expand((t-w)^2+4);
                                t^2 - 2 t w + 4 + w^2
> factor(Basis('union'(SandNrels,ThetaElim,{th2^2-1}),plex(f,g,th1,t
h3,th2,u,v,t,w,d)));
[(d-1)^2(d+1)^2,(d-1)(d+1)(25d+25+128w^2),(d-1)(d+1)(14wd+19w+15t),
-(d+1)(d^2-2d-3+2tw-t^2-w^2),
75t+52w-15td+32wd+15w^3-7d^2w-17d^3w+15t^3-15t^2w-15tw^2,
2-2t^2+8v^2-6w^2-3d+d^3-8tw,
29v-6u+6du-13vd-12wtv+6t^2v+7d^2v+6vw^2+vd^3,-219v+64u+64uw^2
-16w^2vd+53vd+80wtv-32t^2v+64vw^3t-32t^2w^2v+16dwtv-32vw^4-5d^2v
-304vw^2-21vd^3,ut-vt+uw-3vw,6+4uv-3d+d^3-8w^2-8tw,
(d-1)(d^2+2d+2th2-1),(t^2-2tw+4+w^2)(th2-1),(th2-1)(th2+1),
(d+1)(3d^2-2d-5+4th3),28v-6u+4vd-3th3vt^2+6th3vtw-6wtv+3t^2v+6th3u
-24th3v-3th3vw^2-4d^2v+3vw^2-4vd^3,3+d-d^3-d^2-2th2-2th3+2th2th3,th1-1.
(d+1)(7d^2w+10wd+30g-15t-2w),
-3+3d+4w^2-3d^3-4tw+3d^2+16gw+4w^2d-4dwt,
13+19d+4w^2-3d^3-4tw+3d^2+16tg+4w^2d-4dwt,9vt-6w^3v-52vw-48vg
-6vwt^2+12tw^2v+12gu-5wd^2v-6vwd-3vd^3w+9dvt+12uw,
-10g+5t-w-4d^2w+4wd-4d^3w+10th2g-5th2t+5th2w,
13w+13wd-5t-8d^2w-5td-8d^3w+20gth3,2d+1+d^2+4g^2,
-11v-5vd-2vw^2+3d^2v+4wtv-2t^2v+5vd^3+4f]
>

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