

This file contains the computations relevant to the paper "On the classification of non-self-dual modular categories" by Hong and Rowell. Thus covers the case with Galois group (01),(34), $b=0$, $a=-g$ and $y=f$.

```
> with(LinearAlgebra): with(Groebner):
```

Using the Galois argument assuming that (01)(34), the orthogonality relations imply $z=(d-1)$ and $h1=-1/2(d+1)$.

```
> S1:=subs({x=1,y=f,a=-g,b=0},Matrix(5,5,[[1,d,f,g,g],[d,x,y,a,a],[f,y,z,b,b],[g,a,b,h1+I*h2,h1-I*h2],[g,a,b,h1-I*h2,h1+I*h2]]));
```

$$S1 := \begin{bmatrix} 1 & d & f & g & g \\ d & 1 & f & -g & -g \\ f & f & z & 0 & 0 \\ g & -g & 0 & h1+h2I & h1-h2I \\ g & -g & 0 & h1-h2I & h1+h2I \end{bmatrix}$$

```
> C:=Matrix(5,5,[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,0,1],[0,0,0,1,0]]);
```

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

These relations describe the condition that S^2 is proportional to the "charge conjugation matrix."

```
> Srels:=factor(convert(evalm(S1^2-K^2*C),set));
```

```
Srels := {0, f(1+d+z), (-1-2*h1+d)*g, -(-1-2*h1+d)*g, 2*d+f^2-2*g^2, 2*f^2+z^2-K^2, 2*g^2+2*h1^2-2*h2^2, 2*g^2+2*h1^2+2*h2^2-K^2, 1+d^2+f^2+2*g^2-K^2}
```

```
> factor(Basis('union'(Srels, {1+d+z, -1-2*h1+d}), lexdeg([K,d,f,g,z],[h2,h1])));
```

```
[2*h1+2+z, -1-2*h1+d, g^2+h1^2-h2^2, 2+4*h1-2*h2^2+2*h1^2+f^2, (K-2*h2)*(K+2*h2)]
```

```
> slrules:=solve({2+z+2*h1, -1-2*h1+d}, {z,h1});
```

$$slrules := \{h1 = -\frac{1}{2} + \frac{d}{2}, z = -d - 1\}$$

```
> S:=subs(slrules,S1);
```

$$S := \begin{bmatrix} 1 & d & f & g & g \\ d & 1 & f & -g & -g \\ f & f & -d-1 & 0 & 0 \\ g & -g & 0 & \frac{d}{2} - \frac{1}{2} + h2 I & \frac{d}{2} - \frac{1}{2} - h2 I \\ g & -g & 0 & \frac{d}{2} - \frac{1}{2} - h2 I & \frac{d}{2} - \frac{1}{2} + h2 I \end{bmatrix}$$

These relations come down to a single relation among the d, f and g (the other allow elimination of $K, h1$ and $h2$ and z).

> orthrel := {2*d+f^2-2*g^2};

$$\text{orthrel} := \{2d + f^2 - 2g^2\}$$

Observe that if d is an integer, then $d=1$ since (01) interchanges d and $1/d$. this in turn implies that f is an integer dividing $(d+1)=2$, hence $f=1$ or 2 , so that $g^2=(2d+f^2)/2=(2+1)/2$ or $(2+4)/2$. It must be the latter, since g^2 is an algebraic integer. So $g=\sqrt{3}$, $d=1$ and $f=2$ is the only way d is an integer. Note that since (01) interchanges g and $-g/d$, g cannot be an integer, and if f is an integer $f=f/d$ so that $d=1$. We must consider this case eventually.

We use the various symmetries of the $N_{\{i,j\}}^k$ to write down the fusion matrices in terms of just 14 variables. Note that $M4$ is just the transpose of $M3$.

> M1:=Matrix([[0,1,0,0,0],[1,n1,n2,n3,n3],[0,n2,n4,n5,n5],[0,n3,n5,n6,n7],[0,n3,n5,n7,n6]]);

$$M1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & n1 & n2 & n3 & n3 \\ 0 & n2 & n4 & n5 & n5 \\ 0 & n3 & n5 & n6 & n7 \\ 0 & n3 & n5 & n7 & n6 \end{bmatrix}$$

> M2:=Matrix([[0,0,1,0,0],[0,n2,n4,n5,n5],[1,n4,n8,n9,n9],[0,n5,n9,n10,n11],[0,n5,n9,n11,n10]]);

$$M2 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & n2 & n4 & n5 & n5 \\ 1 & n4 & n8 & n9 & n9 \\ 0 & n5 & n9 & n10 & n11 \\ 0 & n5 & n9 & n11 & n10 \end{bmatrix}$$

> M3:=Matrix([[0,0,0,0,1],[0,n3,n5,n7,n6],[0,n5,n9,n11,n10],[1,n6,n10,n12,n13],[0,n7,n11,n14,n12]]);

$$M3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & n3 & n5 & n7 & n6 \\ 0 & n5 & n9 & n11 & n10 \\ 1 & n6 & n10 & n12 & n13 \\ 0 & n7 & n11 & n14 & n12 \end{bmatrix}$$

The matrices must commute, giving a set of diophantine equations.

```
> comrels := `minus` ( `union` ( convert ( evalm ( M1 & * M2 - M2 & * M1 ), set ), convert ( evalm ( M1 & * M3 - M3 & * M1 ), set ), convert ( evalm ( M2 & * M3 - M3 & * M2 ), set ), convert ( evalm ( M3 & * Transpose ( M3 ) - Transpose ( M3 ) & * M3 ), set ) ), { 0 } );
```

```
comrels := { n12 - n13, n13 - n12, n3 n6 + n5 n10 + n7 n13 - n3 n7 - n5 n11 - n7 n14,
n3 n7 + n5 n11 + n7 n14 - n3 n6 - n5 n10 - n7 n13,
n6 n5 + n9 n10 + n11 n13 - n7 n5 - n9 n11 - n11 n14,
n7 n5 + n9 n11 + n11 n14 - n6 n5 - n9 n10 - n11 n13, 1 + n6^2 + n10^2 + n13^2 - n7^2 - n11^2 - n14^2,
1 + n1 n4 + n2 n8 + 2 n3 n9 - n2^2 - n4^2 - 2 n5^2, n2^2 + n4^2 + 2 n5^2 - 1 - n1 n4 - n2 n8 - 2 n3 n9,
n3^2 + n5^2 + 2 n7 n6 - n1 n7 - n2 n11 - n3 n12 - n3 n14,
n5^2 + n9^2 + 2 n11 n10 - n4 n7 - n8 n11 - n9 n12 - n9 n14,
n7^2 + n11^2 + n14^2 - 1 - n6^2 - n10^2 - n13^2,
n1 n7 + n2 n11 + n3 n12 + n3 n14 - n3^2 - n5^2 - 2 n7 n6,
n4 n7 + n8 n11 + n9 n12 + n9 n14 - n5^2 - n9^2 - 2 n11 n10,
n1 n5 + n2 n9 + n3 n10 + n3 n11 - n3 n2 - n5 n4 - n6 n5 - n7 n5,
n2 n5 + n4 n9 + n5 n10 + n5 n11 - n3 n4 - n5 n8 - n6 n9 - n7 n9,
n2 n6 + n4 n10 + n5 n13 + n5 n12 - n3 n5 - n5 n9 - n6 n10 - n7 n11,
n2 n7 + n4 n11 + n5 n12 + n5 n14 - n3 n5 - n5 n9 - n6 n11 - n7 n10,
n3 n2 + n5 n4 + n6 n5 + n7 n5 - n1 n5 - n2 n9 - n3 n10 - n3 n11,
n3 n4 + n5 n8 + n6 n9 + n7 n9 - n2 n5 - n4 n9 - n5 n10 - n5 n11,
n3 n5 + n5 n9 + n6 n10 + n7 n11 - n2 n6 - n4 n10 - n5 n13 - n5 n12,
n3 n5 + n5 n9 + n6 n11 + n7 n10 - n2 n7 - n4 n11 - n5 n12 - n5 n14,
n3 n6 + n5 n10 + n6 n13 + n7 n12 - n3 n7 - n5 n11 - n6 n12 - n7 n14,
n3 n7 + n5 n11 + n6 n12 + n7 n14 - n3 n6 - n5 n10 - n6 n13 - n7 n12,
n6 n5 + n9 n10 + n10 n13 + n11 n12 - n7 n5 - n9 n11 - n10 n12 - n11 n14,
n7 n5 + n9 n11 + n10 n12 + n11 n14 - n6 n5 - n9 n10 - n10 n13 - n11 n12,
1 + n1 n6 + n2 n10 + n3 n13 + n3 n12 - n3^2 - n5^2 - n6^2 - n7^2,
1 + n4 n6 + n8 n10 + n9 n13 + n9 n12 - n5^2 - n9^2 - n10^2 - n11^2,
n3^2 + n5^2 + n6^2 + n7^2 - 1 - n1 n6 - n2 n10 - n3 n13 - n3 n12,
```

```

n5^2+n9^2+n10^2+n11^2-1-n4 n6-n8 n10-n9 n13-n9 n12}

```

```

> indets(comrels);

```

```

{n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9}

```

The commutation relations alone leave 6 degrees of freedom.

```

> HilbertDimension(comrels, tdeg(n1, n10, n11, n12, n13, n14, n2, n3,
n4, n5, n6, n7, n8, n9));

```

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The Characteristic polynomials of M1, M2 and M3 are computed in two ways. First from the knowledge of the eigenvalues.

```

> ch1:=collect(expand(product('(X-S[2,i]/S[1,i])', i=1..5)), X);

```

```

ch2:=collect(expand(product('(X-S[3,i]/S[1,i])', i=1..5)), X);

```

```

ch3:=collect(expand(product('(X-S[4,i]/S[1,i])', i=1..5)), X);

```

$$ch1 := -1 + X^5 + \left(-\frac{1}{d} + 1 - d\right)X^4 + \left(-\frac{1}{d} - d\right)X^3 + \left(\frac{1}{d} + d\right)X^2 + \left(\frac{1}{d} - 1 + d\right)X$$

$$ch2 := X^5 + \left(\frac{1}{f} - \frac{f}{d} + \frac{d}{f} - f\right)X^4 + \left(-2 - d + \frac{f^2}{d} - \frac{1}{d}\right)X^3 + \left(f + \frac{f}{d}\right)X^2$$

$$ch3 := X^5 + \left(-\frac{d}{g} + \frac{g}{d} + \frac{1}{g} - g\right)X^4 + \left(\frac{h2^2}{g^2} - \frac{g^2}{d} - 2 + \frac{d^2}{4g^2} + \frac{1}{d} + d + \frac{1}{4g^2} - \frac{d}{2g^2}\right)X^3$$

$$+ \left(\frac{3d}{4g} + g - \frac{h2^2}{g} - \frac{3}{4g} + \frac{1}{4dg} + \frac{h2^2}{dg} - \frac{d^2}{4g} - \frac{g}{d}\right)X^2 + \left(\frac{1}{2} - \frac{1}{4d} - \frac{d}{4} - \frac{h2^2}{d}\right)X$$

```

>

```

set m=(d+1)/f which in an integer.

```

> simplify({subs({m=(d+1)/f}, coeff(ch2, X, 4)+coeff(ch2, X, 2)-m), subs({
m=(d+1)/f}, subs({X=-m}, ch2))});

```

{0}

The form of these polynomials imply further relations.

```

> simplify({coeff(ch1, X, 0)+1, coeff(ch2, X, 0), coeff(ch2, X, 1), coeff(ch3
, X, 0), coeff(ch1, X, 4)+coeff(ch1, X, 1), coeff(ch1, X, 3)+coeff(ch1, X, 2)
});

```

{0}

```

> factor(ch1); factor(ch2); factor(ch3);

```

$$\frac{(X-1)(X+1)^2(dX-1)(X-d)}{d}$$

d

$$\frac{X^2(X-f)(d+1+fX)(dX-f)}{df}$$

df

$$\frac{X(X-g)(dX+g)(4X^2g^2-4gXd+4gX-2d+1+d^2+4h2^2)}{4dg^2}$$

4dg²

```

> p1:=CharacteristicPolynomial(M1, X);

```

```

p2:=CharacteristicPolynomial(M2, X);

```

```

p3:=CharacteristicPolynomial(M3, X);

```

```

> factor(p1); factor(p2);

```

```

(-n6 + n7 + X)(2 n5^2 - n4 n7 - n4 n6 - X^3 n4 - X^3 n6 - 2 X^2 n5^2 + X^2 n6 n4 - X^2 + X^4 + X n4
+ X^2 n1 n4 + X^2 n1 n6 + X^2 n4 n7 + X^2 n1 n7 + 2 X n1 n5^2 + 2 X n3^2 n4 + X n6 n2^2 + X n2^2 n7
+ X n6 + X n7 - 4 X n2 n3 n5 - X n6 n1 n4 - X n1 n4 n7 - X^3 n7 - X^3 n1 - 2 X^2 n3^2 - X^2 n2^2)
(X - n10 + n11)(2 n5^2 - n2 n11 - n2 n10 - 2 X^2 n5^2 - X^3 n10 - 2 X^2 n9^2 - X^3 n8 + X n10
+ X^2 n10 n8 - X^2 + X^4 + X^2 n8 n11 + X^2 n2 n10 + X^2 n2 n8 + X^2 n2 n11 + 2 X n2 n9^2 + X n10 n4^2
+ X n4^2 n11 + X n11 - 4 X n4 n5 n9 - X n10 n2 n8 - X n2 n8 n11 - X^3 n11 - X^3 n2 - X^2 n4^2
+ X n2 + 2 X n5^2 n8)

```

Observe that the only linear terms of p1 are (X-1) and (X+1), while the linear terms of p2 are X and (X+m) (provided d is not integral)

we may use this later, but for now we avoid using m.

```

> solve(X+n11-n10=(X+m)) ; solve(X+n11-n10=X) ;
      {X=X, m=-n10+n11, n10=n10, n11=n11}
      {X=X, n10=n11, n11=n11}
> mlinrels:={ (n10-n11)*(n10-n11+m) };
      mlinrels := {(n10 - n11)(m + n10 - n11)}
> linrels:={ (n6-n7)^2-1 };
      linrels := {(n6 - n7)^2 - 1}

```

The relations implied by the coefficients of the Characteristic polynomials.

```

> chrels:=factor({coeff(p1,X,0)+1,coeff(p2,X,0),coeff(p2,X,1),coeff(p3,X,0),coeff(p1,X,4)+coeff(p1,X,1),coeff(p1,X,3)+coeff(p1,X,2)});
chrels := {(n10 - n11)(-2 n5^2 + n2 n11 + n2 n10), n4 n6^2 - n4 n7^2 - 2 n5^2 n6 + 2 n5^2 n7 + 1,
n7^2 n9 - 2 n11 n5 n7 + n14 n5^2 - n14 n3 n9 + n11^2 n3, -n10^2 - 4 n5 n4 n9 n11 + 4 n10 n4 n5 n9
+ n11^2 n4^2 + n10^2 n2 n8 + 2 n5^2 - 2 n5^2 n8 n10 - n10^2 n4^2 + 2 n11 n5^2 n8 - 2 n2 n10
- 2 n9^2 n2 n10 - n11^2 n2 n8 + 2 n9^2 n11 n2 + n11^2, -2 n6 - n4 - n1 - 4 n2 n3 n5 n7
+ 4 n2 n3 n5 n6 - n6^2 + n7^2 - 2 n1 n5^2 n6 - 2 n4 n6 + n7^2 n2^2 - n6^2 n2^2 + n1 n4 n6^2
+ 2 n1 n5^2 n7 - n1 n4 n7^2 + 2 n5^2 + 2 n3^2 n7 n4 - 2 n3^2 n4 n6, n1 n4 - 2 n3^2 + 2 n1 n6 + 2 n4 n6
- n2^2 - 2 n5^2 - n7^2 + n6^2 - 1 + 2 n6 n2^2 + n4 n7^2 + 2 n3^2 n6 + 2 n3^2 n4 - 4 n2 n3 n5 - 2 n6 n1 n4
- n4 n6^2 + n4 + 2 n6 - n1 n6^2 - 2 n3^2 n7 + n1 n7^2 + 2 n1 n5^2 - 2 n5^2 n7 + 2 n5^2 n6}
> nops(chrels) ;

```

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these involve the integer $m=z/f=(d+1)/f$.

```

> intchar:=Vector([1,1,-m,0,0]);
> intcharrels:='union'(convert(evalm(M1*intchar-intchar),set),convert(evalm(M2*intchar+m*intchar),set),convert(evalm(M3*intchar),set),convert(evalm(Transpose(M3)*intchar),set));
intcharrels := {
  0, n1 - n2 m, n3 - n5 m, n5 - n9 m, n7 - n11 m, 1 + n6 - n10 m, n2 - n4 m + m, 1 + n4 - n8 m - m^2
}
> mrels:='union'({coeff(p2,X,4)+coeff(p2,X,2)-m,subs({X=-m},p2)},mli

```

nrels, intcharrels);

```
mrels := {0, (n10 - n11)(m + n10 - n11), n1 - n2 m, n3 - n5 m, n5 - n9 m, n7 - n11 m,
1 + n6 - n10 m, n2 - n4 m + m, 1 + n4 - n8 m - m^2, -m^5 - (2 n10 + n8 + n2) m^4
+ (2 n5^2 + 2 n9^2 + n11^2 - n10^2 - 2 n8 n10 - 2 n2 n10 + 1 + n4^2 - n2 n8) m^3 - (-n2 - 2 n10
+ 2 n11 n5^2 - 2 n5^2 n8 + 4 n4 n5 n9 - 2 n2 n9^2 + 2 n10 n2 n8 - 2 n10 n4^2 + 2 n9^2 n11
- 2 n10 n5^2 - 2 n10 n9^2 - n11^2 n8 - n11^2 n2 + n8 n10^2 + n2 n10^2) m^2 + (-2 n5^2 + 2 n2 n10
+ n10^2 - n11^2 - 2 n11 n5^2 n8 + 4 n5 n4 n9 n11 - 4 n10 n4 n5 n9 - 2 n9^2 n11 n2 + 2 n5^2 n8 n10
+ 2 n9^2 n2 n10 + n11^2 n2 n8 - n10^2 n2 n8 - n11^2 n4^2 + n10^2 n4^2) m + 2 n11 n5^2 - 2 n10 n5^2
- n11^2 n2 + n2 n10^2, -n8 + 2 n5^2 n8 - 4 n4 n5 n9 + n11^2 n8 - 2 n9^2 n11 + 2 n10 n5^2 - n2 n10^2
+ 2 n10 n4^2 + n11^2 n2 + 2 n2 n9^2 - 2 n10 n2 n8 + 2 n10 n9^2 - n8 n10^2 - 2 n11 n5^2 - m}
```

We take all of the relations together and process them. .

```
> allNrels := subs({n8=u, n11=v, n13=t, n14=w}, 'union'(linrels, comrels, ch
rels, mrels));
```

```
> factor(Basis(allNrels, lexdeg([w, v, t, u, n1, n6, n9, n10, n12, n2,
n3, n4, n5, n7], [m]))) ;
```

```
[n3 - n5 m, n2 - n4 m + m, -1 - n4 + 2 n10 m, -n5 + n9 m, -n4 + 1 + 2 n6, n1 - n4 m^2 + m^2,
-1 - n4 + u m + m^2, t - n12, -n7 + v m, (2 + m^2)(n7 + 1)(1 - 2 n7 + n4), n7(-2 n10 + m + u),
-1 - n4 + 2 n5^2 + 2 n7^2 + 3 n7 - 3 n4 n7,
n5 + n5 n4 - 2 n7 n5 - 2 n7 w m + n4 m^2 n5 - m^2 n5 + 2 n7 m n12 - 2 n7 m^2 n5,
-1 - n4 + 3 n7 + 2 n7^2 - 3 n4 n7 + 2 m^2 - 2 n4 m^2 + 4 n5 n12 m,
2 n5 n10 - 2 n7 n9 - n5 m + 2 n7 n12 - 2 n7 w - 2 n7 n5 m + m n5 n4,
-m + n5 n9 - 2 n5 n12 + n4 m, -2 n7 n9 + 2 n7 n12 - 2 n7 w + n5 u - 2 n7 n5 m + m n5 n4,
-n7 n9 + n5 v, -3 m - n4 n7 m - 2 n5 n12 + n4 m + 2 n5 w + n7 m + 2 n7^2 m,
(1 - 2 n7 + n4)(n4 - 3 - 2 n7), n5 + 2 m n12 n4 - 2 w m - 2 n7 w m - n5 n4 - 3 m^2 n5
+ 2 n7 m n12 - n7 m^2 n5 - w m^3 - m^4 n5 + n12 n4 m^3,
n4 m + m + n10 + n4 n7 m - n7 m + n4 n10 - 2 n7 v - 2 n7^2 m,
n9 + n4 n9 - 2 n7 n9 - n5 m + 2 n7 n12 - 2 n7 w - 2 n7 n5 m + m n5 n4,
3 n4 m + 3 m + u + 2 n4 n7 m - 2 n7 m - 4 n7 v - 4 n7^2 m + n4 u, v - 2 n7 n10 + n4 v,
-n12 + w + 3 n5 m - 2 n7 n12 + 2 n7 w + 2 n7 n5 m - m n5 n4 + w n4 - n12 n4,
-3 + n4 - 4 n9 n12 + 2 n9^2 + 4 n10^2 - 2 u n10,
-5 - n4 + n7 + 2 n7^2 - n4 n7 + 8 n10^2 + 2 n9 w - 4 u n10 - 2 n9 n12,
1 + n4 + 2 n7 - 4 v n10 + 2 u v - 4 n10^2 + 2 u n10, -1 - n4 + 2 n7^2 - 2 n10^2 + n7 - n4 n7 + 2 v^2,
n5 + 2 v w - 2 v n12 - n5 n4 + 2 n7 n5 - 2 n9 n10 + 2 n9 v,
n4 + n7 + 2 w^2 + 2 n7^2 - n4 n7 - 2 n12^2 - 3, n5 + 2 n7 n5 - n5 n7^2 + 2 m n12 n4 - 3 w m - n7 w m
- 2 n5 n4 - 3 m^2 n5 + n7 m n12 - n7 m^2 n5 - m n12 + 2 n12 n7^2 m, 2 n12 - 2 n7 n9 + 2 n7 n12
```

$$\begin{aligned}
&+ 2 n 7^2 n 9 - 2 n 5 m - 2 n 7 w - 4 n 12 n 7^2 - 3 w m^2 + n 7 n 5 m - 3 m^3 n 5 - m n 5 n 4 + 3 n 12 n 4 m^2 \\
&+ 2 n 12 n 4, -n 7 (n 7 + 1) (-2 n 7 m + 4 n 10 + n 4 m + m - 4 v), -4 n 7 n 12 + n 7 m^3 n 5 + n 7 m^2 w \\
&+ 5 n 5 m + 4 n 7 w - 4 n 12 n 7^2 - 3 w m^2 + 6 n 7 n 5 m - m^3 n 5 + 2 n 12 n 4 m^2 - n 7^2 n 5 m + 4 w n 7^2 \\
&- n 12 n 7 m^2 + 2 n 7^2 m^2 w - 2 m n 5 n 4 - m^2 n 12, 2 n 5 - n 5 n 4 + n 7 n 5 - 3 n 5 n 7^2 + n 4 n 5 n 7 \\
&+ 2 m n 12 n 4 - 3 w m + n 7 w m - m^2 n 5 - n 7 m n 12 + n 7 m^2 n 5 - m n 12 + 2 n 7^2 m w, \\
&n 7 (-3 n 4 m - 8 n 7 n 10 + 8 v + 5 m + n 4 n 7 m - n 7 m - 2 n 7^2 m + 8 n 5 n 12), n 7 n 12 n 4 + n 12 \\
&- 2 n 7 n 12 + n 5 m + n 7 w - 3 n 12 n 7^2 - w m^2 - m^3 n 5 + n 12 n 4 m^2 + w n 7^2 + n 12 n 4 - m n 5 n 4 \\
&+ 2 n 7 n 5 m, \\
&4 n 12 n 7 w + 2 n 4 - 5 n 7 + 2 - 3 n 7^2 + 3 n 4 n 7 + 2 n 7^3 - n 4 n 7^2 - 2 m^2 - 4 n 12^2 n 7 + 2 n 4 m^2, \\
&-2 n 7^2 m w + n 7 m n 12 - 3 n 5 - 2 n 5 n 4 + 6 n 7 n 5 + 8 n 9 n 10 - 2 n 10 n 12 + n 5 n 7^2 + 8 n 10 n 7 n 9 \\
&+ 6 m n 12 n 4 - 8 v n 12 n 7 - 12 n 9 v + 8 v n 12 + 3 n 12 u - 4 w m - n 7 w m + 2 n 10 w - 2 u n 9 \\
&+ u w - 7 m^2 n 5 + 4 m n 12 - 8 n 10 n 12 n 7 - n 7 m^2 n 5, \\
&-1 - n 4 - n 7 + 2 n 7 n 10 v + 2 n 7^2 + n 4 n 7 - 4 n 9 n 12 n 7 - 2 n 10^2 - 2 v n 10 + 2 n 7 n 10^2, 2 n 7^2 m w \\
&- n 7 m n 12 + 3 n 5 - 2 n 5 n 4 + 6 n 7 n 5 - 8 n 9 n 10 + 6 n 10 n 12 - n 5 n 7^2 + 2 m n 12 n 4 + 4 n 9 v \\
&- 3 n 12 u - 4 w m + n 7 w m + 2 n 10 w + 2 u n 9 - u w - m^2 n 5 - 4 m n 12 - 4 n 10 n 12 n 7 \\
&+ 4 w n 7 n 10 + n 7 m^2 n 5, 2 n 7^2 m w - n 7 m n 12 + 3 n 5 - 6 n 5 n 4 + 10 n 7 n 5 - 8 n 9 n 10 \\
&+ 10 n 10 n 12 - n 5 n 7^2 + 10 m n 12 n 4 - 8 v n 12 n 7 - 4 n 9 v + 8 v n 12 - 3 n 12 u - 12 w m + n 7 w m \\
&+ 6 n 10 w + 2 u n 9 - u w - 9 m^2 n 5 - 4 m n 12 - 8 n 10 n 12 n 7 + 8 v n 7 n 9 + n 7 m^2 n 5, -2 m \\
&- 3 n 4 n 7 m - 8 n 10 n 7^2 - 2 n 4 m + 8 n 7 v + 4 n 5 n 12 + 9 n 7 m + n 7^2 m + 3 n 4 n 7^2 m \\
&+ 4 n 12 n 4 n 5 - 6 n 7^3 m, -4 n 7^2 m w + 2 n 7 m n 12 + 16 n 12^2 n 5 - 4 n 5 n 4 + 4 n 7 n 5 + 8 n 9 n 10 \\
&+ 2 n 10 n 12 + 2 n 5 n 7^2 - 4 m n 12 n 4 - 8 v n 12 n 7 - 4 n 9 v + 8 v n 12 + n 12 u - 3 w m - 2 n 7 w m \\
&+ 6 n 10 w - 2 u n 9 - u w - 6 m^2 n 5 + 11 m n 12 - 8 n 10 n 12 n 7 - 2 n 7 m^2 n 5, -8 n 12 w + 13 \\
&+ 13 n 4 - 2 n 7 - 13 n 7^2 - 8 n 7^2 m^2 - 4 n 4 n 7 + 8 n 9 n 12 - 16 n 10^2 + 6 n 7^3 - 5 n 4 n 7^2 - 2 m^4 n 4 \\
&+ 6 m^2 + 16 v n 10 + 8 u n 10 - 16 n 7 n 10^2 + 2 m^4 + 8 n 12^2 n 4 + 2 n 4 m^2 + 4 n 12^2 n 4 m^2 \\
&- 4 n 12 w m^2 + 8 n 9 n 12 n 7, -16 n 10 + 4 u - 4 m - 3 n 4 n 7 m - 8 n 5 n 12 + 8 n 9 v n 12 + 8 n 7 v \\
&- 8 n 12 n 9 n 10 + 5 n 7 m + n 4 n 7^2 m + 7 n 7^2 m - 2 n 7^3 m, -4 w n 10 n 12 - 16 n 10 + 4 u - 3 m \\
&+ 2 w m n 12 + 2 u w n 12 - 3 n 4 n 7 m - 12 n 5 n 12 + 6 m n 12^2 - 12 n 10 n 12^2 + n 4 m + 8 n 7 v \\
&+ 8 n 12 n 9 n 10 - 4 n 12 u n 9 + 3 n 7 m + 6 n 7^2 m + 6 n 12^2 u, -4 n 10 - 8 v + 2 u - 7 m - 5 n 4 n 7 m \\
&+ 8 n 7 n 10 - 8 n 5 n 12 - n 4 m + 8 n 7 v + 8 u n 10^2 + 5 n 7 m + 10 n 7^2 m - 16 n 10^3, -4 n 10 n 12 u \\
&+ 2 n 9 - 2 n 12 - 2 w + 2 n 7 n 9 - 2 n 5 m - w m^2 - 8 n 10^2 n 9 + 4 u n 10 n 9 + n 7 n 5 m - m^3 n 5 \\
&- m n 5 n 4 + n 12 n 4 m^2 + 8 n 10^2 n 12, -2 n 7 n 9 + 2 n 7 n 12 + 2 n 9 v n 10 - n 5 m - 2 n 7 w \\
&- 2 n 10^2 w - 2 n 10^2 n 9 - 2 n 7 n 5 m + m n 5 n 4 + 2 n 10^2 n 12, -44 n 10 - 24 v + 14 u + 4 n 10 u^2
\end{aligned}$$

$$\begin{aligned}
& -25 m - 17 n_4 n_7 m + 24 n_7 n_{10} - 44 n_5 n_{12} + 8 m n_{12}^2 - 16 n_{10} n_{12}^2 + n_4 m + 32 n_7 v \\
& + 8 n_{12} n_9 n_{10} - 4 n_{12} u n_9 + 17 n_7 m + 34 n_7^2 m + 8 n_{12}^2 u - 16 n_{10}^3, 2 n_{10} n_{12} u - 2 n_9 \\
& + 2 n_{12} + 2 w + 2 n_7 n_9 - 2 n_7 n_{12} + n_5 m + 2 n_7 w + w m^2 - 4 n_{10}^2 w + n_7 n_5 m + m^3 n_5 \\
& - n_{12} n_4 m^2 - 4 n_{10}^2 n_{12} + 2 u w n_{10}, -4 n_{10} n_{12} u + 2 m^2 n_{12} + 6 n_9 - 6 n_{12} - 6 w + 2 n_7 n_9 \\
& + 2 u^2 n_9 + 4 n_7 n_{12} - 10 n_5 m - 4 n_7 w - 3 w m^2 - 8 n_{10}^2 n_9 - n_7 n_5 m - 3 m^3 n_5 - 2 n_{12} u^2 \\
& - m n_5 n_4 + 3 n_{12} n_4 m^2 + 16 n_{10}^2 n_{12}, 4 n_{10} n_{12} u - m^2 n_{12} - 6 n_9 + 6 n_{12} + 6 w + 6 n_7 n_9 \\
& - 8 n_7 n_{12} + 6 n_5 m + 8 n_7 w + 2 w m^2 - 4 n_{10}^2 w + 5 n_7 n_5 m + 3 m^3 n_5 + n_{12} u^2 - m n_5 n_4 \\
& - 3 n_{12} n_4 m^2 - 12 n_{10}^2 n_{12} + u^2 w, 24 n_{12} w - 14 - 14 n_4 - 36 n_7 - 5 n_7^2 + 8 n_7^2 m^2 + 40 n_4 n_7 \\
& - 24 n_9 n_{12} + 40 n_{10}^2 + 8 n_{12}^2 - 5 n_7^3 - 2 n_7^4 + n_4 n_7^3 + 11 n_4 n_7^2 + 8 n_7^2 n_{10}^2 - 12 m^2 \\
& - 56 v n_{10} - 24 u n_{10} + 56 n_7 n_{10}^2 - 16 n_{12}^2 n_4 + 4 n_4 m^2 - 16 n_{12}^2 n_7^2 - 48 n_9 n_{12} n_7, \\
& -5 n_7^3 m n_4 + 24 w n_{10} n_{12} + 20 m^3 - 40 n_{10} - 16 v + 12 u - 38 m - 40 w m n_{12} - 50 n_4 n_7 m \\
& + 16 n_7 n_{10} + 24 n_{10} n_7^2 + 32 n_7 n_{10}^3 - 48 v n_{12}^2 n_7 + 24 n_{10} n_{12}^2 - 18 n_4 m - 8 n_7 v \\
& - 32 n_{12} n_9 n_{10} + 66 n_7 m + 40 m n_{12}^2 n_4 + 3 n_4 n_7^2 m + 55 n_7^2 m - 20 n_4 m^3 - 16 n_{10} n_{12}^2 n_7 \\
& + 10 n_7^4 m - 11 n_7^3 m - 32 v n_{10}^2 + 16 n_{12}^2 v, 4 + 4 n_4 + 3 n_7 + n_7^2 - 5 n_4 n_7 - 24 n_{10}^2 \\
& + 4 n_{12}^2 + 2 n_7^3 - 16 n_{12}^2 n_{10}^2 - n_4 n_7^2 + 2 m^2 + 8 v n_{10} + 16 u n_{10} - 8 n_7 n_{10}^2 - 2 u^2 \\
& + 4 n_{12}^2 n_4 + 8 n_{12}^2 u n_{10} + 8 n_9 n_{12} n_7, 2 + 2 n_4 + 2 n_7 + 3 n_7^2 - 6 n_4 n_7 + 3 n_7^3 - 2 n_7^4 \\
& - 8 n_{12}^2 n_{10}^2 + n_4 n_7^3 - n_4 n_7^2 + 8 v n_{10} - 8 n_{10}^2 + 4 u n_{10} + 8 n_{12} n_{10}^2 w - 8 n_7 n_{10}^2 \\
& + 16 n_9 n_{12} n_7, -56 w n_{10} n_{12} - 8 m^3 + 976 n_{10} + 224 v - 272 u + 325 m - 8 u^3 + 238 n_4 n_7 m \\
& - 224 n_7 n_{10} + 736 n_5 n_{12} + 128 n_{10} n_{12}^4 - 16 v n_{12}^2 n_7 - 312 m n_{12}^2 + 664 n_{10} n_{12}^2 \\
& - 59 n_4 m - 528 n_7 v - 272 n_{12} n_9 n_{10} + 136 n_{12} u n_9 - 64 m n_{12}^4 - 252 n_7 m + 16 m n_{12}^2 n_4 \\
& - 7 n_4 n_7^2 m - 483 n_7^2 m - 296 n_{12}^2 u + 32 n_{12}^3 u n_9 + 48 n_{10} n_{12}^2 n_7 + 14 n_7^3 m \\
& - 64 n_{12}^3 n_9 n_{10} + 64 n_{10}^3 - 48 n_{12}^2 v - 64 n_{12}^4 u, 24 n_{12} w - 8 - 8 n_4 + 68 n_7 + 126 n_7^2 \\
& - 156 n_4 n_7 - 24 n_9 n_{12} - 32 n_{10}^2 + 80 n_{12}^2 - 32 n_{10}^4 + 2 u^4 + 87 n_7^3 - 5 n_7^4 - 128 n_{12}^2 n_{10}^2 \\
& + 3 n_4 n_7^3 - 42 n_4 n_7^2 - 8 n_{12}^4 m^2 - 36 m^2 + 56 v n_{10} - 16 u n_{10} + 32 n_9 n_{12}^3 n_7 \\
& - 32 n_{10}^2 n_{12}^2 n_7 - 56 n_7 n_{10}^2 - 2 m^4 + 32 u^2 + 56 n_{12}^2 n_4 + 8 n_{12}^4 u^2 - 2 n_7^5 + 16 n_{12}^4 n_4 \\
& + 20 n_4 m^2 - 32 m^2 n_{12}^2 + 32 n_{12}^2 u^2 - 32 n_{12}^2 n_7^2 + 16 n_{12}^4 - 32 n_{12}^4 n_{10}^2 + n_7^4 n_4 \\
& + 144 n_9 n_{12} n_7 + 32 n_{12}^2 v n_{10}]
\end{aligned}$$

[m>0 an integer implies:.

```

> factor(Basis('union'(allNrels, {1-2*n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n
12-n5+n7*n5, n5*m-n7*n12+n7*w}), plex(n1, n6, n9, n12, n2, n3, n4, w,
v, n10, t, u, n5, n7, m)));

```

$$\begin{aligned}
& [n_5^2 + 2 n_7 - 2 n_7^2, -2 n_7 + m^2 + u m, -(n_7 - 1)(-4 n_7 t + 3 n_5 m + n_5 u), \\
& n_7 + n_5 m t - n_7^2 + m^2 - n_7 m^2, -n_7(-3 m - u + u n_7 - 2 n_5 t + 3 n_7 m),
\end{aligned}$$

$6n7 + 2un5t - u^2n7 - 3m^2 + u^2 - 6n7^2 + 3n7m^2, -n7 + n10m, -n7(-2n10 + m + u),$
 $-n5(-2n10 + m + u), -2n7 + m^2 + 2un10 - u^2, -4n7 + 4n10^2 + m^2 - u^2,$
 $-6n7m + 3m + 2n10 + 4v - u + 4n5t - 2un7, -n5 + tm + wm + n7n5 + m^2n5 - 2n7mt,$
 $n5m - n7t + n7w, 2n7m - 2m - n5t + n5w,$
 $-3n5m + 2n5u + 2tm^2 + n7n5m - u^2t + m^3n5 - 2n7m^2t + u^2w, 8m - 8u + 32n10 - 19n7m$
 $- 7un7 + 22n5t + 2m^3 - 5n7^2m + 2t^2m - n7^2u - 4ut^2 + 12n10t^2 - 2n7m^3 - 8n7mt^2$
 $+ 8t^3n5 - 4t^2un7 + 2uwt,$
 $n5 + 5tm + n7n5 + 2n10t + m^2n5 - 8n7mt + 4n5t^2 - 2un7t - uw + 4n10w,$
 $2n7 - 2 + w^2 - t^2, 1 - 2n7 + n4, n3 - n5m, n2 - 2n7m + 2m, n12 - t,$
 $-2t - 2w - 3n5m + 4n7t - n5u + 2n9, -n7 + 1 + n6, 2m^2 - 2n7m^2 + n1]$

assuming $n7=0$ gives $m=0$.

```
> factor(Basis('union'(allNrels, {1-2*n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5, n5*m-n7*n12+n7*w, n7}), plex(n1, n6, n9, n12, n2, n3, n4, w, v, n10, t, u, n5, n7, m)));
```

$[m^2, n7, n5m, n5^2, um, n5u, u^2, n10m, n5n10, un10, n10^2, 4n5t - u + 4v + 2n10 + 3m,$
 $wm - n5 + tm, -2m - n5t + n5w,$
 $4m - 4u + 16n10 + 11n5t + t^2m - 2ut^2 + 6n10t^2 + 4t^3n5 + uwt,$
 $n5 + 5tm + 4n5t^2 + 2n10t - uw + 4n10w, w^2 - 2 - t^2, n4 + 1, n3, n2 + 2m, n12 - t, n9 - w - t,$
 $n6 + 1, n1]$

Thus we may assume $n7$ not zero since then we have $m=0$.

```
> factor(Basis('union'(allNrels, {1-2*n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5, n5*m-n7*n12+n7*w, -3*m-u-2*n5*t+u*n7+3*n7*m, -2*n10+u+m}), plex(n1, n6, n9, n12, n2, n3, n4, w, v, n10, t, u, n5, n7, m)));
```

$[n5^2 + 2n7 - 2n7^2, -2n7 + m^2 + um, -(n7 - 1)(-4n7t + 3n5m + n5u),$
 $3m + u - un7 + 2n5t - 3n7m, 2n10 - m - u, -m + 2v - u,$
 $-n5 + tm + wm + n7n5 + m^2n5 - 2n7mt, n5m - n7t + n7w, -m + u + n7m - un7 + 2n5w,$
 $3n5 - 2tm - n7n5 - m^2n5 + 2n7mt + uw - ut, 2n7 - 2 + w^2 - t^2, 1 - 2n7 + n4, n3 - n5m,$
 $n2 - 2n7m + 2m, n12 - t, -2t - 2w - 3n5m + 4n7t - n5u + 2n9, -n7 + 1 + n6,$
 $2m^2 - 2n7m^2 + n1]$

finally, we solve for all but w, v, t, u .

```
> intrels:=factor(Basis('union'(allNrels, {1-2*n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5, n5*m-n7*n12+n7*w, -3*m-u-2*n5*t+u*n7+3*n7*m, -2*n10+u+m}), lexdeg([m, n1, n6, n9, n10, n12, n2, n3, n4, n5, n7], [u, v, t, w])));
```

$intrels := [2 + t^2 - w^2 + 2uv - 4v^2, (-w + t)(-3vw - vt + ut + uw),$
 $-(-w + t)(t^3 + t^2w + 2t - 2tv^2 - tw^2 + 2wv^2 + 2w - w^3), 2n7 - 2 + w^2 - t^2, vw - vt + n5,$
 $-1 + n4 + w^2 - t^2, 2n3 + 2w + t^2w - w^3 - 2t - t^3 + tw^2, 2wtv + n2 - t^2v - vw^2, n12 - t, n10 - v,$

$$t^3 + t^2 w - t w^2 - w^3 + 2 n_9 + 2 w v^2 - 2 t v^2, w^2 - t^2 + 2 n_6,$$

$$2 n_1 + 4 t w + 2 w t^3 - 2 t w^3 - 2 w^2 + w^4 - 2 t^2 - t^4, m - 2 v + u]$$

> rules1 := 'union' ({n8=u, n11=v, n13=t, n14=w}, solve({seq(intrels[i], i=4 ..14)}));

$$rules1 := \{m = -u + 2v, n1 = t^2 - 2tw + w^2 + \frac{1}{2}t^4 - wt^3 + tw^3 - \frac{1}{2}w^4, n10 = v, n11 = v, n12 = t,$$

$$n13 = t, n14 = w, n2 = vw^2 - 2w tv + t^2 v, n3 = t - w + \frac{1}{2}t^3 - \frac{1}{2}t^2 w - \frac{1}{2}t w^2 + \frac{1}{2}w^3,$$

$$n4 = 1 + t^2 - w^2, n5 = -vw + vt, n6 = \frac{t^2}{2} - \frac{w^2}{2}, n7 = 1 + \frac{t^2}{2} - \frac{w^2}{2}, n8 = u,$$

$$n9 = \frac{1}{2}w^3 + \frac{1}{2}t w^2 - \frac{1}{2}t^2 w - w v^2 - \frac{1}{2}t^3 + t v^2, t = t, u = u, v = v, w = w\}$$

> indets(subs(rules1, 'union'(allNrels, {1-2*n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5, n5*m-n7*n12+n7*w, -3*m-u-2*n5*t+u*n7+3*n7*m, -2*n10+u+m}))) ;

$$\{t, u, v, w\}$$

> agb := factor(Basis(subs(rules1, 'union'(allNrels, {1-2*n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5, n5*m-n7*n12+n7*w, -3*m-u-2*n5*t+u*n7+3*n7*m, -2*n10+u+m}))), plex(u, v, t, w));

$$agb := [-(w+t)(t^3 + t^2 w + 2t - 2t v^2 - t w^2 + 2w v^2 + 2w - w^3),$$

$$(-w+t)(-3vw - vt + ut + uw), 2 + t^2 - w^2 + 2uv - 4v^2]$$

> M1s := subs(rules1, M1); M2s := subs(rules1, M2); M3s := subs(rules1, M3);

M1s :=

$$[0, 1, 0, 0, 0]$$

$$\left[1, t^2 - 2tw + w^2 + \frac{1}{2}t^4 - wt^3 + tw^3 - \frac{1}{2}w^4, vw^2 - 2w tv + t^2 v,$$

$$t - w + \frac{1}{2}t^3 - \frac{1}{2}t^2 w - \frac{1}{2}t w^2 + \frac{1}{2}w^3, t - w + \frac{1}{2}t^3 - \frac{1}{2}t^2 w - \frac{1}{2}t w^2 + \frac{1}{2}w^3 \right]$$

$$[0, vw^2 - 2w tv + t^2 v, 1 + t^2 - w^2, -vw + vt, -vw + vt]$$

$$\left[0, t - w + \frac{1}{2}t^3 - \frac{1}{2}t^2 w - \frac{1}{2}t w^2 + \frac{1}{2}w^3, -vw + vt, \frac{t^2}{2} - \frac{w^2}{2}, 1 + \frac{t^2}{2} - \frac{w^2}{2} \right]$$

$$\left[0, t - w + \frac{1}{2}t^3 - \frac{1}{2}t^2 w - \frac{1}{2}t w^2 + \frac{1}{2}w^3, -vw + vt, 1 + \frac{t^2}{2} - \frac{w^2}{2}, \frac{t^2}{2} - \frac{w^2}{2} \right]$$

M2s :=

$$[0, 0, 1, 0, 0]$$

$$[0, vw^2 - 2w tv + t^2 v, 1 + t^2 - w^2, -vw + vt, -vw + vt]$$

$$\begin{aligned}
& \left[1, 1+t^2-w^2, u, \frac{1}{2}w^3+\frac{1}{2}tw^2-\frac{1}{2}t^2w-wv^2-\frac{1}{2}t^3+tv^2, \right. \\
& \left. \frac{1}{2}w^3+\frac{1}{2}tw^2-\frac{1}{2}t^2w-wv^2-\frac{1}{2}t^3+tv^2 \right] \\
& \left[0, -vw+vt, \frac{1}{2}w^3+\frac{1}{2}tw^2-\frac{1}{2}t^2w-wv^2-\frac{1}{2}t^3+tv^2, v, v \right] \\
& \left[0, -vw+vt, \frac{1}{2}w^3+\frac{1}{2}tw^2-\frac{1}{2}t^2w-wv^2-\frac{1}{2}t^3+tv^2, v, v \right] \\
M_{3s} := & \begin{bmatrix} 0, & 0, & 0, & 0, & 1 \\ 0, & t-w+\frac{1}{2}t^3-\frac{1}{2}t^2w-\frac{1}{2}tw^2+\frac{1}{2}w^3, & -vw+vt, & 1+\frac{t^2}{2}-\frac{w^2}{2}, & \frac{t^2}{2}-\frac{w^2}{2} \\ 0, & -vw+vt, & \frac{1}{2}w^3+\frac{1}{2}tw^2-\frac{1}{2}t^2w-wv^2-\frac{1}{2}t^3+tv^2, & v, & v \\ 1, & \frac{t^2}{2}-\frac{w^2}{2}, & v, & t, & t \\ 0, & 1+\frac{t^2}{2}-\frac{w^2}{2}, & v, & w, & t \end{bmatrix}
\end{aligned}$$

>

The unique positive character is psi0.

> psi0:=Vector([1,d,f,g,g]): psi2:=subs({m=(d+1)/f},intchar):

> psirels:=numer('minus'('union'(convert(evalm(M1s*psi0-psi0[2]*psi0),set),convert(evalm(M2s*psi0-psi0[3]*psi0),set),convert(evalm(M3s*psi0-psi0[4]*psi0),set),convert(evalm(M1s*psi2-psi2[2]*psi2),set),convert(evalm(M2s*psi2-psi2[3]*psi2),set),convert(evalm(M3s*psi2-psi2[4]*psi2),set)),{0})));

$$\begin{aligned}
psirels := & \{2f+t^2f-w^2f-2vd-2v, 2+d t^2-d w^2+2vf+4tg-2g^2, \\
& 2d+d t^2-d w^2+2vf+2wg+2tg-2g^2, f v w^2-2f w t v+f t^2 v-d t^2-t^2+d w^2+w^2, \\
& 2f^2+t^2f^2-w^2f^2-ufd-uf-d^2-2d-1, \\
& d v w^2-2d w t v+d t^2 v+f+t^2f-w^2f-2g v w+2g v t-d f, \\
& 2t f-2w f+f t^3-f t^2 w-f t w^2+f w^3+2d v w+2v w-2d v t-2v t, \\
& -2d v w+2d v t+f w^3+f t w^2-f t^2 w-2f w v^2-f t^3+2f t v^2+4v g-2f g, \\
& 1+d+d t^2-d w^2+u f+g w^3+g t w^2-g t^2 w-2g w v^2-g t^3+2g t v^2-f^2, \\
& 2d t-2d w+t^3 d-t^2 w d-t w^2 d+w^3 d-2v w f+2v t f+2g t^2-2g w^2+2g-2d g, 2 t^2 f \\
& -4 t w f+2 w^2 f+t^4 f-2 w t^3 f+2 t w^3 f-w^4 f-2 d t^2 v-2 t^2 v+4 d w t v+4 w t v-2 d v w^2 \\
& -2 v w^2,-2 v w f+2 v t f+t^3 d+t^3+t^2 w d+t^2 w-t w^2 d-t w^2-w^3 d-w^3+2 w v^2 d+2 w v^2
\end{aligned}$$

$$-2tv^2d - 2tv^2, 2 + 2dt^2 - 4dtw + 2dw^2 + dt^4 - 2dwt^3 + 2dtw^3 - dw^4 + 2fvw^2$$

$$-4fwtv + 2ft^2v + 4tg - 4wg + 2gt^3 - 2gt^2w - 2gtw^2 + 2gw^3 - 2d^2\}$$

later we will consider the case $d=1, f=2$ and $g=\sqrt{3}$.

```
psirels2:=subs({d=1,f=2,g=sqrt(3)},numer('minus'('union'(convert(evalm(M1*psi0-psi0[2]*psi0),set),convert(evalm(M2*psi0-psi0[3]*psi0),set),convert(evalm(M3*psi0-psi0[4]*psi0),set),convert(evalm(M1*psi2-psi2[2]*psi2),set),convert(evalm(M2*psi2-psi2[3]*psi2),set),convert(evalm(M3*psi2-psi2[4]*psi2),set)),{0})))));
```

```
psirels2 := {2n1 - 2n2, 2n3 - 2n5, 4n4 - 4n8, 2n5 - 2n9, 2n7 - 2n11, 2 + 2n6 - 2n10,
n1 + 2n2 + 2n3*sqrt(3), 2n2 - 2n4 + 2, -3 + n4 + 2n8 + 2n9*sqrt(3), n2 + 2n4 + 2n5*sqrt(3) - 2,
-2 + n6 + 2n10 + n12*sqrt(3) + n13*sqrt(3), n3 + 2n5 + n6*sqrt(3) + n7*sqrt(3) - sqrt(3),
n5 + 2n9 + n10*sqrt(3) + n11*sqrt(3) - 2*sqrt(3), n7 + 2n11 + n14*sqrt(3) + n12*sqrt(3) - 3}
```

```
> indets(psirels2);
```

```
{n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9}
```

```
> dlbasis:=factor(Basis(psirels2,tdeg(n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9)));
```

```
dlbasis := [-3 + 2n9*sqrt(3) + 3n8, -1 + n9*sqrt(3) + n7 + n6, n5 - n9, -3 + 2n9*sqrt(3) + 3n4, -n9 + n3,
2n9*sqrt(3) + 3n2, 2*sqrt(3) - 3n9 - 2n7*sqrt(3) - n14 + n13, n14 + n12 + n7*sqrt(3) - sqrt(3), -n7 + n11,
-2 + n9*sqrt(3) + n7 + n10, 2n9*sqrt(3) + 3n1]
```

```
> solve(convert(dlbasis,set));
```

$$\{n1 = -\frac{2n9\sqrt{3}}{3}, n10 = 2 - n9\sqrt{3} - n7, n11 = n7, n12 = -n14 - n7\sqrt{3} + \sqrt{3},$$

$$n13 = -2\sqrt{3} + 3n9 + 2n7\sqrt{3} + n14, n14 = n14, n2 = -\frac{2n9\sqrt{3}}{3}, n3 = n9, n4 = 1 - \frac{2n9\sqrt{3}}{3},$$

$$n5 = n9, n6 = 1 - n9\sqrt{3} - n7, n7 = n7, n8 = 1 - \frac{2n9\sqrt{3}}{3}, n9 = n9\}$$

since these are integers, we see that if $d=1$ we get the following rules:

```
> dlrules:={n1=0,n9=0,n8=1,n7=1,n10=1,n12=0,n14=0,n13=0,n2=0,n4=1,n5=0,n6=0,n3=0,n11=1};
```

```
dlrules := {n1 = 0, n10 = 1, n11 = 1, n12 = 0, n13 = 0, n14 = 0, n2 = 0, n3 = 0, n4 = 1, n5 = 0,
n6 = 0, n7 = 1, n8 = 1, n9 = 0}
```

```
> subs(dlrules,[M1,M2,M3]);
```

$$\left[\left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right], \left[\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right], \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array} \right] \right]$$

```
> map(Eigenvalues,%);
```

$$\left[\begin{array}{c} \left[\begin{array}{c} -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{array} \right] \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 2 \\ 2 \end{array} \right] \left[\begin{array}{c} 0 \\ \sqrt{3} \\ -\sqrt{3} \\ I \\ -I \end{array} \right] \end{array} \right]$$

We combine the nontrivial relations obtained so far.

```
> SandNrels := `union` (psirels, convert(agb, set), orthrel);
```

```
SandNrels := {(-w+t)(-3vw-vt+ut+uw),
```

$$-(-w+t)(t^3+t^2w+2t-2tv^2-tw^2+2wv^2+2w-w^3), 2d+f^2-2g^2,$$

$$2+t^2-w^2+2uv-4v^2, 2f+t^2f-w^2f-2vd-2v, 2+dt^2-dw^2+2vf+4tg-2g^2,$$

$$2d+dt^2-dw^2+2vf+2wg+2tg-2g^2, fvw^2-2fwtv+ft^2v-dt^2-t^2+dw^2+w^2,$$

$$2f^2+t^2f^2-w^2f^2-ufd-uf-d^2-2d-1,$$

$$dvw^2-2dwtv+dt^2v+f+t^2f-w^2f-2gvw+2gvt-df,$$

$$2tf-2wf+ft^3-ft^2w-ftw^2+fw^3+2dvw+2vw-2dvt-2vt,$$

$$-2dvw+2dvt+fw^3+ftw^2-ft^2w-2fww^2-ft^3+2ftv^2+4vg-2fg,$$

$$1+d+dt^2-dw^2+uf+gw^3+gtw^2-gt^2w-2gww^2-gt^3+2gtv^2-f^2,$$

$$2dt-2dw+t^3d-t^2wd-tw^2d+w^3d-2vwf+2vtf+2gt^2-2gw^2+2g-2dg, 2t^2f$$

$$-4twf+2w^2f+t^4f-2wt^3f+2tw^3f-w^4f-2dt^2v-2t^2v+4dwtv+4wtv-2dvw^2$$

$$-2vw^2, -2vwf+2vtf+t^3d+t^3+t^2wd+t^2w-tw^2d-tw^2-w^3d-w^3+2wv^2d+2wv^2$$

$$-2tv^2d-2tv^2, 2+2dt^2-4dtw+2dw^2+dt^4-2dwt^3+2dtw^3-dw^4+2fvw^2$$

$$-4fwtv+2ft^2v+4tg-4wg+2gt^3-2gt^2w-2gtw^2+2gw^3-2d^2}$$

```
> indets(SandNrels);
```

{d, f, g, t, u, v, w}

```
> SandNbasis := factor(Basis(SandNrels, plex(d, f, g, u, v, t, w)));
```

```
SandNbasis := [ -(-w+t)(t^3+t^2w+2t-2tv^2-tw^2+2wv^2+2w-w^3),
```

$$-(t^3+t^2w+2t-2tv^2-tw^2+2wv^2+2w-w^3)(2v^2+1+t^2+2tw-3w^2),$$

$$2tv^3-t^3v-2wv^3-t^2wv+w^2tv+w^3v+ut-3vt+uw-5vw, 2+t^2-w^2+2uv-4v^2,$$

$$-g(t^3+t^2w+2t-2tv^2-tw^2+2wv^2+2w-w^3), -2-w^2+t^2+2g^2-gw^3-gt^3-6tg$$

$$-2wg-2wt^3+2tw^3+2w^4-4v^2w^2+4wtv^2-4v^2-2t^2w^2+gtw^2+gt^2w,$$

$$2f-4v-w^2f+t^2f-2gvt+2gvw,$$

$$-tg-wg-wt^3+tw^3-w^2+w^4+t^2-2v^2w^2+2wtv^2-2v^2-t^2w^2+vf,$$

$$2+uf-tg-3wg-2wt^3+2tw^3-2w^2+2w^4+2t^2-4v^2w^2+4wtv^2-4v^2-2t^2w^2, 3vw$$

```

+vt-w^3v-w^2tv+t^2wv+t^3v+2wv^3-2tv^3-2vg-vgw^2+2gvvtw-t^2gv-wf-tf
+fg,-w^2+t^2-4v^2+2w^4+2tw^3-2t^2w^2-4v^2w^2-2wt^3-4wg+4wtv^2-4tg-gw^3
+gtw^2+gt^2w-gt^3+f^2,-tg-1+wg+d]

```

The next step is to introduce the relations involving the roots of unity theta_i. The first goal is to show that th2 satisfies a nontrivial degree 3 (or less) polynomial in Q[d].

```

> psith0:=Vector([1,d*th1,f*th2,g*th3,g*th3]):
> Thetarels:=simplify(subs({},[th1^2*S[2,2]-(sum('M1[k,2]*psith0[k]',k=1..5)),th1*th2*S[2,3]-(sum('M1[k,3]*psith0[k]',k=1..5)),th1*th3*S[2,4]-(sum('M1[k,4]*psith0[k]',k=1..5)),th3*th2*S[4,3]-(sum('M2[k,4]*psith0[k]',k=1..5)),th2^2*S[3,3]-(sum('M2[k,3]*psith0[k]',k=1..5)),th3^2*(S[4,4]+S[4,5])-(sum('(M3[k,4]+M3[k,5])*psith0[k]',k=1..5))])));
Thetarels := [th1^2 - 1 - n1 d th1 - n2 f th2 - 2 n3 g th3, th1 th2 f - n2 d th1 - n4 f th2 - 2 n5 g th3,
               -th1 th3 g - n3 d th1 - n5 f th2 - n6 g th3 - n7 g th3, -n5 d th1 - n9 f th2 - n10 g th3 - n11 g th3,
               -th2^2 d - th2^2 - 1 - n4 d th1 - n8 f th2 - 2 n9 g th3,
               th3^2 d - th3^2 - 1 - d th1 n7 - d th1 n6 - f th2 n11 - f th2 n10 - 2 g th3 n12 - g th3 n13 - g th3 n14]

```

we immediately find that th1=1.

```

> factor(Basis('union'(SandNrels,convert(subs(rules1,Thetarels),set)),tdeg(th1,u,v,t,w,d,th2,th3,f,g)))[7];
               g th3 (th1 - 1)
> Thetarels1:=factor( numer(subs({th1=1},Thetarels)) );
Thetarels1 := [-n1 d - n2 f th2 - 2 n3 g th3, f th2 - n2 d - n4 f th2 - 2 n5 g th3,
               -g th3 - n3 d - n5 f th2 - n6 g th3 - n7 g th3, -n5 d - n9 f th2 - n10 g th3 - n11 g th3,
               -th2^2 d - th2^2 - 1 - n4 d - n8 f th2 - 2 n9 g th3,
               th3^2 d - th3^2 - 1 - d n7 - d n6 - f th2 n11 - f th2 n10 - 2 g th3 n12 - g th3 n13 - g th3 n14]

```

```

> indets(Thetarels1);
      {d, f, g, n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9, th2, th3}
> map(degree,Thetarels1,th3);
      [1, 1, 1, 1, 1, 2]
> solve(Thetarels1[2],th3);

```

$$-\frac{-fth2 + n2d + n4fth2}{2n5g}$$

This is valid unless n5=0, which implies d=1 (which we consider below).

```

> factor(Basis(subs(rules1,'union'(SandNrels,{n5})),plex(u,t,v,w,f,g,d)));
[(d-1)(d+1), 2d+f^2-2g^2, dg^2+4wg-g^2-2+2d, 4v-f-df, dg+2t-2dw-g,
  u(d-1), 2f+2df-fg^2-dfg^2-2u+2g^2u, 2d-g^2+uf-dg^2+2, -w(df+f-u)]

```

We substitute back in to eliminate th3 and take at the numerators of the resulting rational functions, which give us our new relations.

```

> Thetarels2:=factor( numer(subs({th3=solve(Thetarels1[2],th3)},Theta

```

```
rels1))) ;
```

```
Thetarels2 := [-n1 d n5 - n2 f th2 n5 - n3 f th2 + n3 n2 d + n3 n4 f th2, 0, -f th2 + n2 d + n4 f th2
- 2 n3 d n5 - 2 n5^2 f th2 - n6 f th2 + n6 n2 d + n6 n4 f th2 - n7 f th2 + n7 n2 d + n7 n4 f th2,
-2 n5^2 d - 2 n9 f th2 n5 - f th2 n10 + n10 n2 d + n10 n4 f th2 - f th2 n11 + n11 n2 d + n11 n4 f th2,
-th2^2 d n5 - th2^2 n5 - n5 - n4 d n5 - n8 f th2 n5 - n9 f th2 + n9 n2 d + n9 n4 f th2,
4 n12 n5 g^2 n4 f th2 + 2 n13 n5 g^2 n4 f th2 + 2 n14 n5 g^2 n4 f th2 + 2 f^2 th2^2 n4 - n4^2 f^2 th2^2
+ d f^2 th2^2 - 4 f th2 n10 n5^2 g^2 - 4 f th2 n11 n5^2 g^2 - 2 n2 d n4 f th2 + 2 n2 d^2 n4 f th2
- 4 n12 n5 g^2 f th2 + 4 n12 n5 g^2 n2 d - 2 n13 n5 g^2 f th2 + 2 n13 n5 g^2 n2 d - 2 n14 n5 g^2 f th2
+ 2 n14 n5 g^2 n2 d - f^2 th2^2 - n2^2 d^2 - 4 n5^2 g^2 - 4 d n7 n5^2 g^2 - 4 d n6 n5^2 g^2 + 2 f th2 n2 d
- 2 f th2 n2 d^2 - 2 d f^2 th2^2 n4 + d n4^2 f^2 th2^2 + n2^2 d^3]
> Thetarels2red := factor(Reduce(subs(rules1, Thetarels2), SandNbasis, plex(d, f, g, u, v, t, w, th2))) ;
```

```
> map(degree, Thetarels2red, th2) ;
```

```
[-∞, -∞, -∞, -∞, 2, 2]
```

```
> factor([coeff(Thetarels2red[5], th2, 2), coeff(Thetarels2red[5], th2, 1), coeff(Thetarels2red[5], th2, 0)]) ;
```

```
[-2 v (-w + t) (t g + 2 - w g), 4 t^3 v + 12 v t + 4 t^2 w v + 2 t^2 g v - 8 t v^3 - 4 w^2 t v + 8 w v^3 - 4 w^3 v
- 4 g v t w - 4 t f + 2 v g w^2 + 4 v w - 4 w f, -2 v (-w + t) (t g + 2 - w g)]
```

provided $v(t-w)$ is not zero, $th2$ satisfies a nonzero degree 2 polynomial (observe that g cannot be rational). The following shows that if $v(t-w)=0$, then $d=1$, which we will consider separately.

```
> factor(Basis('union'(SandNrels, {v*(t-w)}), plex(f, g, u, v, t, w, d))) ;
```

```
[(d-1)(d+1), (-w+t)(d+1), -w^2+t^2-d+1, v(d-1), v(-w+t), w(4v^2+d+1), u(d-1),
```

```
w(u-4v), ut-4vw, 2uv+1-4v^2+d, dg+2t-2dw-g, w(dw+g-t),
```

```
1-d+tg+dw^2-tw, 1-3d+2g^2-4v^2+4dw^2-4tw, f+df-4v, v(-2v+f),
```

```
uf+1-4v^2+d, fg-2vg-tf-wf+4vw, 1-d-4tw-4v^2+4dw^2+f^2]
```

```
> ThetaElim := subs(rules1, 'union'(convert(Thetarels, set), {th1-1}))) ;
```

```
ThetaElim := {th1-1,
```

$$-(v w + v t) d th1 - \left(\frac{1}{2} w^3 + \frac{1}{2} t w^2 - \frac{1}{2} t^2 w - w v^2 - \frac{1}{2} t^3 + t v^2 \right) f th2 - 2 v g th3,$$

$$th1 th2 f - (v w^2 - 2 w t v + t^2 v) d th1 - (1 + t^2 - w^2) f th2 - 2 (-v w + v t) g th3, th1^2 - 1$$

$$- \left(t^2 - 2 t w + w^2 + \frac{1}{2} t^4 - w t^3 + t w^3 - \frac{1}{2} w^4 \right) d th1 - (v w^2 - 2 w t v + t^2 v) f th2$$

$$- 2 \left(t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3 \right) g th3, -th1 th3 g$$

$$- \left(t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3 \right) d th1 - (-v w + v t) f th2 - \left(\frac{t^2}{2} - \frac{w^2}{2} \right) g th3$$

```

-  $\left(1 + \frac{t^2}{2} - \frac{w^2}{2}\right) g \text{ th3}, -\text{th2}^2 d - \text{th2}^2 - 1 - (1 + t^2 - w^2) d \text{ th1} - u f \text{ th2}$ 
-  $2 \left(\frac{1}{2} w^3 + \frac{1}{2} t w^2 - \frac{1}{2} t^2 w - w v^2 - \frac{1}{2} t^3 + t v^2\right) g \text{ th3},$ 
 $\text{th3}^2 d - \text{th3}^2 - 1 - d \text{ th1} \left(1 + \frac{t^2}{2} - \frac{w^2}{2}\right) - d \text{ th1} \left(\frac{t^2}{2} - \frac{w^2}{2}\right) - 2 f \text{ th2} v - 3 g \text{ th3} t - g \text{ th3} w$ 
> indets(ThetaElim);
{d, f, g, t, th1, th2, th3, u, v, w}
> indets(allNrels2);
{m, n1, n10, n12, n2, n3, n4, n5, n6, n7, n9, t, u, v, w}
first suppose d=1. This forces th3=0.
> factor(Basis(subs('union'({d=1, f=2, g=sqrt(3)}, d, rules), convert(ThetaElim, set)), tdeg(th1, th3, th2)));
[th2, th3, th1 + 1]
> with(numtheory):
Since [Q[d,f,g]:Q]=2, th2 satisfies a cyclotomic polynomial of degree 1,2 or 4.
> invphi(4); invphi(2);
[5, 8, 10, 12]
[3, 4, 6]
> bases1:=i->factor(Basis('union'(SandNrels, ThetaElim, {cyclotomic(i, th2)}), plex(d, f, g, th1, th3, th2, u, v, t, w)))[1];
bases1 := i -> factor(Groebner-Basis(
'union'(SandNrels, ThetaElim, {numtheory:-cyclotomic(i, th2)}),
plex(d, f, g, th1, th3, th2, u, v, t, w)))_1
> map(bases1, invphi(4));
[(22201 - 19008 w^6 + 20736 w^8 - 201456 w^4 - 65532 w^2)(t^2 - 2 t w + 4 + w^2),
(-36481 - 89856 w^6 + 10368 w^8 + 159984 w^4 + 19704 w^2)(t^2 - 2 t w + 4 + w^2),
(-58081 - 302400 w^6 + 103680 w^8 + 187920 w^4 + 75660 w^2)(t^2 - 2 t w + 4 + w^2),
(6 w^2 + 18 w + 13)(6 w^2 - 18 w + 13)(288 w^4 + 24 w^2 - 121)(t^2 - 2 t w + 4 + w^2)]
> map(bases1, invphi(2));
[(9 w^2 + 4)(t^2 - 2 t w + 4 + w^2), (-1 + 60 w^2 + 72 w^4)(t^2 - 2 t w + 4 + w^2), t^2 - 2 t w + 4 + w^2]
clearly have no solutions.
> expand((t-w)^2+4);
t^2 - 2 t w + 4 + w^2
finally we see that if th2^2=1, we get d=1.
> factor(Basis('union'(SandNrels, ThetaElim, {th2^2-1}), plex(f, g, th1, th3, th2, u, v, t, w, d)));
[(d-1)^2 (d+1)^2, -(d-1)(d+1)(25 d - 25 - 128 w^2), -(d-1)(d+1)(14 w d - 19 w - 15 t),
-(d-1)(d^2 + 2 d - 3 + 2 t w - w^2 - t^2),

```

$$\begin{aligned}
& 4 - 6d + 2tw^3 + 6t^2 - w^4 + 2d^3 - 4tw - 2wt^3 - 2w^2 + t^4, 11 + 29d + 64v^2 + 5d^2 - 16tw^3 \\
& - 16t^2 - 16w^2d + 16w^4 - 13d^3 + 16dwt + 16tw + 16wt^3 - 16t^2w^2, \\
& -29v + 6u + 6du - 13vd + 12wtv - 6t^2v - 7d^2v + vd^3 - 6vw^2, -219v + 64u + 64uw^2 \\
& + 16w^2vd - 53vd + 80wtv - 32t^2v + 64vw^3t - 32t^2w^2v - 16dwtv - 32vw^4 - 5d^2v \\
& + 21vd^3 - 304vw^2, -105vt + 30ut - 15w^3v - 142vw + 15vw^2t + 15tw^2v - 15vt^3 \\
& + 7wd^2v + 32vw d - 17vd^3w - 15dvt + 30uw, 43 + 32uv + 29d + 5d^2 - 16tw^3 - 16w^2d \\
& + 16w^4 - 13d^3 + 16dwt + 16tw + 16wt^3 - 16t^2w^2 - 16w^2, (d+1)(d^2 - 2d - 1 + 2th2), \\
& (t^2 - 2tw + 4 + w^2)(th2 - 1), (th2 - 1)(th2 + 1), (d - 1)(3d^2 + 2d - 5 + 4th3), 28v - 6u - 4vd \\
& - 3th3vt^2 + 6th3v tw - 6w tv + 3t^2v + 6th3u - 24th3v - 3th3vw^2 - 4d^2v + 4vd^3 + 3vw^2 \\
& , -d - d^2 - 2th2 - 2th3 + 2th2th3 + 3 + d^3, th1 - 1, (d - 1)(7d^2w - 10wd + 30g - 15t - 2w), \\
& -3 - 3d + 16gw + 4w^2 - 4w^2d + 3d^2 + 3d^3 + 4dwt - 4tw, \\
& 13 - 19d + 4w^2 + 16tg - 4w^2d + 3d^2 + 3d^3 + 4dwt - 4tw, 9vt - 6w^3v - 52vw - 48vg \\
& - 6vw^2t^2 + 12tw^2v + 12gu - 5wd^2v + 6vwd + 3vd^3w - 9dvt + 12uw, \\
& -10g + 5t - w - 4d^2w - 4wd + 4d^3w + 10th2g - 5th2t + 5th2w, \\
& 13w - 13wd - 5t - 8d^2w + 5td + 8d^3w + 20gth3, 1 - 2d + d^2 + 4g^2, \\
& -11v + 5vd - 2vw^2 + 4w tv + 3d^2v - 2t^2v - 5vd^3 + 4f]
\end{aligned}$$

[>