M409 Exam 2 and Solutions

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Do all problems and show all work. The numbers in parentheses indicate the point value of each problem.

1. Let

\[
A = \begin{pmatrix}
0 & 1 & 0 & -3 \\
1 & 1 & 0 & 2 \\
-1 & 1 & 1 & -3 \\
1 & -1 & 0 & 4
\end{pmatrix}
\]

(a) (10) Find the characteristic and minimal polynomials of \(A\) (hint: there are two distinct eigenvalues).

(b) (20) Apply the Primary Decomposition Theorem to the matrix \(A\) to find a matrix \(B\) that is similar to \(A\) but is in block diagonal form. (hint: since there are two distinct eigenvalues there should be two blocks. The solution is not unique).

0.1. Solution. The characteristic polynomial is \((x - 1)^2(x - 2)^2\), while the minimal polynomial is \((x - 1)(x - 2)^2\). The solution is not unique. The form of the answer is

\[
\begin{pmatrix}
C & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where \(C\) is any matrix similar to \(\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\). (This will probably be the most common solution).

2. (30) Let \(T \in \text{End}(V)\), with \(\dim(V) = n\). Show that \(T\) has a cyclic vector if and only if any \(U \in \text{End}(V)\) that commutes with \(T\) (i.e. \(UT = TU\)) is a polynomial of degree at most \(n - 1\) in \(T\). This means there exists scalars \(\{c_i\}\) so that \(U = \sum c_i T^i\).

0.2. Solution. If \(T\) has a cyclic vector \(\alpha\) then \(B = \{\alpha, T\alpha, \ldots, T^{n-1}\alpha\}\) is a basis for \(V\). Suppose that \(U\) commutes with \(T\) and let \(U\alpha = c_0 \alpha + c_1 T\alpha + \cdots + c_{n-1} T^{n-1}\alpha\).

Claim that \(U = \sum c_i T^i\). Enough to check that this is true on any basis, such as \(B\). But this is clear:

\[
U(T^j \alpha) = T^j U(\alpha) = T^j(\sum_{i=0}^{n-1} c_i T^i \alpha) = (\sum_{i=0}^{n-1} c_i T^i)(T^j \alpha)
\]

Now suppose that \(T\) has no cyclic vector. The cyclic decomposition theorem tells us that \(T\) is similar to a block diagonal matrix:

\[
A = \begin{pmatrix}
M_1 & 0 & \cdots & 0 \\
0 & M_2 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & M_k
\end{pmatrix}
\]

Where \(M_1\) is the companion matrix to the minimal polynomial \(m_T(x)\), and \(M_1\) is the companion matrix to a polynomial \(p_2\) that divides the minimal polynomial. \(T\)
has no cyclic vector, $M_2$ is non-trivial, and $p_2$ is of degree at least 1. The block matrix:

$$B = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & I
\end{pmatrix}$$

obviously commutes with $A$, but since

$$f(A) = \begin{pmatrix}
f(M_1) & 0 & \cdots & 0 \\
0 & f(M_2) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & f(M_k)
\end{pmatrix}$$

if $B = f(A)$ for some polynomial $f$ we have $f(M_1) = 0$ so $m_T(x) | f$ since $m_T(x)$ is the minimal polynomial of $M_1$. But $p_2 | m_T(x)$, so $f(M_i) = 0$ for all $M_i$, forcing $B = 0$. This is a contradiction. □

3. Recall (or take my word for it) that the function $f(x) = e^x$ has Taylor series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ valid for any real number $x$. Let $D$ be any diagonal $m \times m$ matrix and $N$ any nilpotent $m \times m$ matrix. Define $e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!}$ and $e^N$ similarly. In this problem you will show that $e^C$ makes sense (converges for any matrix $C$).

(a) (10) Find a single matrix form for $e^D$ in terms of its diagonal entries.

Solution (using fact about Taylor series above):

$$\begin{pmatrix}
e^{d_1} & 0 & \cdots & 0 \\
0 & e^{d_2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & e^{d_n}
\end{pmatrix}$$

(b) (10) Show that $e^N = \sum_{n=0}^{m} \frac{N^n}{n!}$ so that $e^N$ is well-defined.

Solution: $N$ is nilpotent, so $N^{m+j} = 0$ for all $j \geq 0$. Thus the infinite sum of matrices is actually finite.

(c) (10) Suppose $A$ and $B$ are commuting matrices, and $e^A$ and $e^B$ do make sense (in other words, the infinite series $e^A$ converges to some matrix.) Show that $(e^A)(e^B) = e^{(A+B)}$. Thus $e^{(A+B)}$ makes sense as soon as both $e^A$ and $e^B$ do.

Solution: Just multiply out the terms on each side and compare. Since $A$ and $B$ commute, one may use the binomial theorem to multiply out $(A + B)^n$. A precise proof would just involve induction, but even working out the first few terms is okay.

(d) (10) (Let $P$ be any invertible $m \times m$ matrix, and $A$ any matrix for which $e^A$ makes sense (converges). Show that $e^{(P^{-1}AP)} = P^{-1}e^A P$, so that if $e^A$ makes sense, so does $e^{(P^{-1}AP)}$.}
Solution: $(P^{-1}AP)^k = P^{-1}A^k P$ so the left hand side can be transformed into the right.

The Jordan decomposition theorem implies that any matrix $C$ is similar to a sum $D + N$ where $D$ is diagonal, $N$ is nilpotent, and where $D$ and $N$ commute (see also Section 6.8 Theorem 13). We conclude that $e^C$ is well-defined for any matrix $C$.

**Extra Credit** (5) Let $A$ be a fixed $m \times m$ matrix, and $t$ a real variable. What is $\frac{d}{dt}(e^{At})(0)$ that is, (the derivative of $e^{At}$ at $t = 0$)? The derivative of a matrix is gotten by differentiating each of its entries. A guess will receive no points, you must prove your answer.

The answer is of course $A$. The proof involves computing that $\frac{d}{dt}(e^{At}) = Ae^{At}$ and the observation that $e^{0} = I$ (0 being the zero matrix).

The exam is due in class on Friday, April 15, 2005.