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From Quantum Groups to Unitary Modular Tensor Categories

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Abstract. Modular tensor categories are generalizations of the representation categories of quantum groups at roots of unity axiomatizing the properties necessary to produce 3-dimensional TQFTs. Although other constructions have since been found, quantum groups remain the most prolific source. Recently proposed applications to quantum computing have provided an impetus to understand and describe these examples as explicitly as possible, especially those that are "physically feasible." We survey the current status of the problem of producing unitary modular tensor categories from quantum groups, emphasizing explicit computations.

1. Introduction

We outline the development of the theory of modular tensor categories from quantum groups with an eye towards new applications to quantum computing that motivate our point of view. In this article, we take *quantum group* to mean the "classical" q-deformation of the universal enveloping algebra of a simple complex finite dimensional Lie algebra as in the book by Lusztig [**L**], rather than the broader class of Hopf algebras the term sometimes describes.

1.1. Background. The representation theory of quantum groups has proven to be a useful tool and a fruitful source of examples in many areas of mathematics. The general definition of a quantum group (as a Hopf algebra) was given around 1985 by Drinfeld [**D**] and independently Jimbo [**Ji**] as a method for finding solutions to the quantum Yang-Baxter equation. These solutions led to new representations of Artin's braid group $\mathcal{B}_n$ and connections to link invariants. In fact, specializations of the famous polynomial invariants of Jones [**J**], the six-authored paper [**HOMFLY**] and Kauffman [**Kf**] have been obtained in this way. Reshetikhin and Turaev [**RT**] used this connection to derive invariants of 3-manifolds from *modular Hopf algebras*, examples of which can be found among quantum groups at roots of unity (see [**RT**] and [**TW1**], much simplified by constructions in [**A**]). When Witten [**Wi**] introduced the notion of a topological quantum field theory (TQFT)

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relating ideas from quantum field theory to manifold invariants, non-trivial examples were immediately available from the constructions in [**RT**] (after reconciling notation). Modular Hopf algebras were replaced by the more general framework of modular tensor categories (MTCs) by Turaev [**T1**] (building on definitions in [**Mc**] and [**JS**]), axiomatizing the conditions necessary (and sufficient, see [**T2**], Introduction) to construct 3-dimensional TQFTs.

Aside from the quantum group approach to MTCs, there are several other general constructions. Representation categories of Hopf algebra doubles of finite group algebras are examples of MTCs that are often included with quantum groups in the more general discussion of Hopf algebra constructions. A geometric construction using link invariants and tangle categories was introduced in [**T2**], advanced by Turaev and Wenzl in [**TW2**] and somewhat simplified by Blanchet and Beliakova in [**BB**]. However, all examples that have been carried out lead to MTCs also obtainable from quantum groups. Yet another construction of MTCs from representations of vertex operator algebras has recently appeared in a paper by Huang [**Hu**]. See Subsection 3.2 for further discussion of these approaches.

Although it is expected that there are non-trivial examples of MTCs that do not arise from Hopf algebras (*e. g.* quantum groups and finite group algebras), none have been rigorously produced. This is probably due to the highly advanced state of the theory of representations of quantum groups at roots of unity provided by the pioneering work of many including Lusztig ([**L**]) and Andersen and his co-authors ([**A**], [**AP**] [**APW**]). The description of the MTCs derived from quantum groups can be understood with little more than a firm grasp on the theory of representations of simple finite-dimensional Lie algebras found in Humphrey's book [**Hm**] or any other introductory text.

The purpose of this paper is two-fold: to survey what is known about the modularity and unitarity of categories arising from quantum groups at roots of unity, and to give useful combinatorial tools for explicit computations in these categories. For more in-depth developments the reader is directed to two references: 1) Bakalov and Kirillov's text ([**BK**], Sections 1.3 and 3.3) contains concise constructions and examples of quantum group MTCs, and 2) Sawin's paper ([**S2**]) gives a thorough treatment of the representation theoretic details necessary to construct MTCs from quantum groups. The modularity results described below partially overlap with Section 6 of [**S2**].

1.2. Motivation. There are two fairly well-known motivations for studying MTCs. They are applications to low-dimensional topology (see [**T2**]), and conformal field theory (see [**Hu**] and references therein).

Recently, an application of unitary MTCs to quantum computing has been proposed by Freedman and Kitaev and advanced in the series of papers ([**FKW**], [**FKLW**], [**FLW**] and [**FNSWW**]). Their topological model for quantum computing has a major advantage over the "classical" qubit model in that errors are corrected on the physical level and so has a higher error threshold. For a very readable introduction to topological quantum computing see [**FKW**]. In this model unitary MTCs play the role of the software, while the hardware is implemented via a quantum physical system and the interface between them is achieved by a 3-dimensional TQFT. The MTCs encode the symmetries of the corresponding physical systems (called topological states, see [**FNSWW**]), and must be unitary by physical considerations.

Aside from the problem of constructing unitary MTCs, there are several open problems currently being studied related to the quantum computing applications. One question is whether the images of the irreducible unitary braid representations (see Remark 2.1) afforded by a unitary MTC are dense in the unitary group. This is related to a *sine qua non* of quantum computation known as *universality.* Progress towards answering this question has been made in [**FLW**] and was extended by Larsen, Wang, and the author in [**LRW**]. Another problem is to prove the conjecture of Z. Wang: *There are finitely many MTCs of a fixed rank* (see Subsection 2.2). This has been verified for ranks 1,2,3 and 4: see [**O1**] and [**O2**] for ranks 2 and 3 respectively, and [**BRSW**] for both ranks 3 and 4. It is with this conjecture in mind that we provide generating functions for ranks of categories in Subsection 4.7.

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2. General Definitions

We give the basic categorical definitions for modular tensor categories, remark on some consequences and describe the crucial condition of unitarity.

2.1. Axioms. In this subsection we outline the axioms for the categories we are interested in. We follow the paper [**T1**], and refer to that paper or the books by Turaev [**T2**] or Kassel [**K**] for a complete treatment.

Let $\mathcal{O}$ be a category defined over a subfield $k \subset \mathbb{C}$. A modular tensor category is a semisimple ribbon Ab-category $\mathcal{O}$ with finitely many isomorphism classes of simple objects satisfying a non-degeneracy condition. We unravel these adjectives with the following definitions.

(1) A **monoidal category** is a category with a tensor product $\otimes$ and an identity object $\mathbb{1}$ satisfying axioms that guarantee that the tensor product is associative (at least up to isomorphism) and that
$$\mathbb{1} \otimes X \cong X \otimes \mathbb{1} \cong X$$
for any object X. See [**Mc**] for details.

(2) A monoidal category has **duality** if there is a dual object X^* for each object X and morphisms
$$b_X : \mathbb{1} \to X \otimes X^*, d_X : X^* \otimes X \to \mathbb{1}$$
satisfying
$$\begin{aligned} (\mathrm{Id}_X \otimes d_X)(b_X \otimes \mathrm{Id}_X) &= \mathrm{Id}_X, \\ (d_X \otimes \mathrm{Id}_{X^*})(\mathrm{Id}_{X*} \otimes b_X) &= \mathrm{Id}_{X^*}. \end{aligned}$$
The duality allows us to define duals of morphisms too: for any $f \in \mathrm{Hom}(X, Y)$ we define $f^* \in \mathrm{Hom}(Y^*, X^*)$ by:
$$f^* = (d_Y \otimes \mathrm{Id}_{X^*})(\mathrm{Id}_{Y^*} \otimes f \otimes \mathrm{Id}_{X^*})(\mathrm{Id}_{Y^*} \otimes b_X).$$

(3) A **braiding** in a monoidal category is a natural family of isomorphisms

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying

$$\begin{aligned} c_{X,Y\otimes Z} &= (\mathrm{Id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{Id}_Z), \\ c_{X\otimes Y,Z} &= (c_{X,Z} \otimes \mathrm{Id}_Y)(\mathrm{Id}_X \otimes c_{Y,Z}). \end{aligned}$$

(4) A **twist** in a braided monoidal category is a natural family of isomorphisms

$$\theta_X : X \to X$$

satisfying:

$$\theta_{X\otimes Y} = c_{Y,X}c_{X,Y}(\theta_X \otimes \theta_Y).$$

(5) In the presence of a braiding, a twist and duality these structures are compatible if

$$\theta_{X^*} = (\theta_X)^*.$$

A braided monoidal category with a twist and a compatible duality is a **ribbon** category.

(6) An *Ab***-category** is a monoidal category in which all morphism spaces are k-vector spaces and the composition and tensor product of morphisms are bilinear.

(7) An *Ab*-category is **semisimple** if it has the property that every object X is isomorphic to a finite direct sum of *simple* objects–that is, objects X_i with $\mathrm{End}(X_i) \cong k$ satisfying the conclusion of Schur's Lemma:

$$\mathrm{Hom}(X_i, X_j) = 0 \quad \text{for } i \neq j.$$

Turaev [**T2**] gives a weaker condition for semisimplicity avoiding direct sums, but we omit it for brevity's sake.

(8) In a ribbon *Ab*-category one may define a k-linear **trace** of endomorphisms. Let $f \in \mathrm{End}(X)$ for some object X. Set:

$$tr(f) = d_X c_{X,X^*}(\theta_X f \otimes \mathrm{Id}_{X^*})b_X$$

where the right hand side is an element of $\mathrm{End}(\mathbb{1}) \cong k$. The value of $tr(\mathrm{Id}_X)$ is called the categorical dimension of X and denoted $\dim(X)$.

(9) A semisimple ribbon *Ab*-category is called a **modular tensor category** if it has finitely many isomorphism classes of simple objects enumerated as $\{X_0 = \mathbb{1}, X_1, \ldots, X_{n-1}\}$ and the so called *S-matrix* with entries

$$S_{i,j} := tr(c_{X_j,X_i} \circ c_{X_i,X_j})$$

is invertible. Observe that S is a symmetric matrix.

2.2. Notation and Remarks. In a semisimple ribbon *Ab*-category $\mathcal{O}$ with finitely many simple classes the set of simple classes generates a semiring over k under $\otimes$ and $\oplus$. This ring is called the *Grothendieck semiring* and denoted $Gr(\mathcal{O})$. If $\{X_0 = \mathbb{1}, X_1, \ldots, X_{n-1}\}$ is a set of representatives of these isomorphism classes, the **rank** of $\mathcal{O}$ is n. The axioms guarantee that we have (using Kirillov's notation [**Ki**]):

$$X_i \otimes X_j \cong \sum_k N_{i,j}^k X_k \tag{2.1}$$

for some $N_{i,j}^k \in \mathbb{N}$. These structure coefficients of $Gr(\mathcal{O})$ are called the *fusion coefficients* of $\mathcal{O}$ and (2.1) is sometimes called a fusion rule. Having fixed an ordering of simple objects as above, the fusion coefficients give us a representation of $Gr(\mathcal{O})$ via $X_i \to N_i$ where N_i, $(N_i)_{k,j} = (N_{i,j}^k)$ is called the *fusion matrix* associated to X_i. If we denote by i^* the index of the simple object X_i^*, the braiding and associativity constraints give us:

$$\begin{aligned} N_{i,j}^k = N_{j,i}^k &= N_{i,k^*}^{j^*} = N_{i^*,j^*}^{k^*}, \\ N_{i,j}^0 &= \delta_{i,j^*}. \end{aligned}$$

It also follows from associativity that the fusion matrices pairwise commute, so that full fusion rules may sometimes be computed just from a single fusion matrix (*i.e.* using a Gröbner basis algorithm).

The first column (and row) of the S-matrix consists of the categorical dimensions of the simple objects, *i.e.* $S_{i,0} = \dim(X_i)$. We denote these dimensions by d_i. We also have that $S_{i,j} = S_{j,i} = S_{i^*,j^*}$. Since the twists $\theta_X \in \mathrm{End}(X)$ for any object X, θ_{X_i} is a scalar map (as X_i is simple). We denote this scalar by θ_i.

Standard arguments show that the entries of the S-matrix are determined by the categorical dimensions, the fusion rules and the twists on these simple classes, giving the following extremely useful formula (see [**BK**]):

$$S_{i,j} = \frac{1}{\theta_i \theta_j} \sum_k N_{i^*,j}^k d_k \theta_k. \tag{2.2}$$

Provided $\mathcal{O}$ is modular the S-matrix determines the fusion rules via the *Verlinde formula* (see [**BK**], and [**Hu**]). To express the formula we must introduce the quantity $D^2 = \sum_i d_i^2$. Then:

$$N_{i,j}^k = \sum_t \frac{S_{i,t} S_{j,t} S_{k^*,t}}{D^2 S_{0,t}}. \tag{2.3}$$

This formula corresponds to the following fact: the columns of the S-matrix are simultaneous eigenvectors for the fusion matrices N_i, and the categorical dimensions are eigenvalues.

Remark 2.1. The braiding morphisms $c_{X,X}$ induce a representation of the braid group $\mathcal{B}_n$ on $\mathrm{End}(X^{\otimes n})$ for any object X via the operators

$$R_i = \mathrm{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{Id}_X^{\otimes n-i-1} \in \mathrm{End}(X^{\otimes n})$$

and the generators σ_i of $\mathcal{B}_n$ act by left composition by R_i.

Remark 2.2. The term "modular" comes from the following fact: if we set $T = (\delta_{i,j}\theta_i)_{ij}$ then the map:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to S, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to T$$

defines a projective representation of the *modular group* $SL(2,\mathbb{Z})$. In fact, by renormalizing S and T one gets an honest representation of $SL(2,\mathbb{Z})$.

2.3. Unitarity. A **Hermitian** ribbon Ab-category has a conjugation:

$$\dagger : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$$

such that $(f^\dagger)^\dagger = f$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ and $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$. On $k \subset \mathbb{C}$, $\dagger$ must also act as the usual conjugation. Furthermore, $\dagger$ must be compatible with the other structures present *i.e.*

$$\begin{aligned}
(c_{X,Y})^\dagger &= (c_{X,Y})^{-1}, \\
(\theta_X)^\dagger &= (\theta_X)^{-1}, \\
(b_X)^\dagger &= d_X c_{X,X^*}(\theta_X \otimes \mathrm{Id}_{X^*}), \\
(d_X)^\dagger &= (\mathrm{Id}_{X^*} \otimes \theta_X^{-1})(c_{X^*,X})^{-1} b_X.
\end{aligned}$$

For Hermitian ribbon Ab-categories the categorical dimensions d_i are always real numbers. If in addition the Hermitian form $(f,g) = tr(fg^\dagger)$ is positive definite on $\mathrm{Hom}(X,Y)$ for any two objects $X, Y \in \mathcal{O}$, the category is called **unitary**, and the categorical dimensions are positive real numbers. If $\mathcal{O}$ is unitary, then the morphism spaces $\mathrm{End}(X)$ are Hilbert spaces with the above form, and the representations

$$\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$$

described in Remark 2.1 are unitary.

3. Constructions

MTCs have been derived in varying degrees of detail from several sources. A very general approach is through representations of quantum groups at roots of unity. We give a very broad outline of how these are obtained and mention a few other sources and constructions.

3.1. MTCs from Quantum Groups. The following construction is now standard, and can be found in more detail in the books by Turaev [**T2**] or Bakalov and Kirillov Jr. [**BK**] (both of which include examples). The procedure is a culmination of the work of many, but the major contributions following those of Drinfeld and Jimbo were from Lusztig (see [**L**]), Andersen and his collaborators ([**APW**],[**A**] and [**AP**]) and Turaev with Reshetikhin ([**RT**]) and Wenzl ([**TW1**]). Let $\mathfrak{g}$ be a Lie algebra from one of the infinite families $ABCD$ or an exceptional Lie algebra of type E, F or G and q a complex number such that q^2 is a primitive ℓth root of unity, where ℓ is greater than or equal to the dual Coxeter number of $\mathfrak{g}$. Let $U = U_q(\mathfrak{g})$ be Lusztig's [**L**] "modified form" of the Drinfeld-Jimbo quantum group specialized at q and denote by $\mathcal{T}$ Andersen's [**A**] subcategory of *tilting modules* over U. A module V is called *tilting* if both V and its dual, V^*, admit Weyl filtrations: *i.e.* sequences

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with each V_i/V_{i-1} a Weyl module. The ratio of the square lengths of a long root to a short root will play an important role in the sequel, so we denote it by the letter m. Observe that $m = 1$ for Lie types ADE, $m = 2$ for Lie types BC and F, and $m = 3$ for Lie type G. It can be shown that $\mathcal{T}$ is a (non-semisimple) ribbon Ab-category (see [**A**] and [**TW1**]). The ribbon structure on $\mathcal{T}$ comes from the (ribbon) Hopf algebra structure on U (see [**ChP**]), *i.e.* the antipode, comultiplication, R-matrix, quantum Casimir *etc.* The set of indecomposable tilting modules with $\dim(X) = 0$ (categorical dimension) generates a tensor ideal $\mathcal{I} \subset \mathcal{T}$, and semisimplicity is recovered by taking the quotient category $\mathcal{F} = \mathcal{T}/\mathcal{I}$. Moreover, the category $\mathcal{F}$ has only finitely many isomorphism classes of simple objects, labelled by the subset of dominant weights (denoted P_+) in the *fundamental alcove*:

$$C_\ell(\mathfrak{g}) := \begin{cases} \{\lambda \in P_+ : \langle \lambda + \rho, \vartheta_0 \rangle < \ell\} & \text{if } m \mid \ell \\ \{\lambda \in P_+ : \langle \lambda + \rho, \vartheta_1 \rangle < \ell\} & \text{if } m \nmid \ell \end{cases}$$

where ϑ_0 is the highest root and ϑ_1 is the highest short root. Here the form $\langle\ ,\ \rangle$ is normalized so that $\langle \alpha, \alpha \rangle = 2$ for *short* roots. While $\mathcal{F}$ is always a semisimple Hermitian ribbon Ab-category with finitely many isomorphism classes of simple objects, the further properties (modularity and unitarity) of $\mathcal{F}$ depend on $\mathfrak{g}$, the divisibility of ℓ by m, and the specific choice of q. We denote the category $\mathcal{F}$ by $\mathcal{C}(\mathfrak{g}, \ell, q)$ to emphasize this dependence. The S-matrices for these categories are well-known. For $\lambda, \mu \in C_\ell(\mathfrak{g})$ we have:

$$S_{\lambda,\mu} = \frac{\sum_{w \in W} \varepsilon(w) q^{2\langle \lambda+\rho, w(\mu+\rho)\rangle}}{\sum_{w \in W} \varepsilon(w) q^{2\langle \rho, w(\rho)\rangle}} \tag{3.1}$$

where ρ is the half sum of the positive roots and $\varepsilon(w)$ denotes the sign of the Weyl group element w.

Remark 3.1. In practice, Formula (2.2) is often more useful than Formula (3.1) for computing the entries of the S-matrix, as computing the twists θ_λ, q-dimensions d_λ (see below) and fusion coefficients $N_{\lambda,\mu}^{\nu}$ (via the quantum Racah formula, see [**AP**] and [**S2**]) is more straightforward than summing over the Weyl group.

The twist coefficients for simple objects are also well known: $\theta_\lambda = q^{\langle \lambda, \lambda + 2\rho \rangle}$, as are the categorical q-dimensions:

$$d_\lambda = \prod_{\alpha \in \Phi_+} \frac{[\langle \lambda + \rho, \alpha \rangle]}{[\langle \rho, \alpha \rangle]}$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and Φ_+ is the set of positive roots.

We note that the fusion coefficients of $\mathcal{C}(\mathfrak{g}, \ell, q)$ only depend on $\mathfrak{g}$ and ℓ. A complete description of the braiding and associativity maps is quite difficult in general; fortunately one is usually content to know they exist, relying on the S-matrix, fusion matrices and twists for most calculations.

Remark 3.2. An issue has recently come to light regarding the explicit fusion rules for these categories. While Andersen-Paradowski [**AP**] showed that for many cases the fusion rules for the truncated tensor product in the category $\mathcal{F}$ are determined from the classical multiplicities by an anti-symmetrization over the affine Weyl group, their proof appeared in a paper that restricted attention to the root lattice. Evidently the first general proof of this “quantum Racah” formula is in the preprint [**S2**].

3.2. Other Constructions. The most direct construction of MTCs comes from the representation category of the semidirect product $D(G) := k[G] \ltimes \mathcal{F}(G)$ of the group algebra of a finite group with its (Hopf algebra) dual and can be found in the book [**BK**]. For example, the representation category of the Hopf algebra $D(S_3)$ is a rank 8 MTC that does not arise from a quantum group construction as outlined above. These MTCs always have integer q-dimensions.

The geometric construction of MTCs alluded to in the introduction is summarized as follows. One starts with a link invariant satisfying a number of mild (but technical) conditions and produces a new category from the category of tangles via an *idempotent completion* of quotients of endomorphism spaces. This produces a semisimple braided category, and if there is explicit information available for the link invariant one can sometimes verify the remaining axioms. This has been carried out for the Jones polynomial (Chapter XII of [**T2**]) and the Kauffman polynomial [**TW2**]. Blanchet and Beliakova [**BB**] gave a complete analysis of the modularity and modularizability of these categories corresponding to BMW algebras–the algebras supporting the Kauffman polynomial. Although the work in [**BB**] eliminated the need to appeal to quantum group characters as in [**TW2**], these constructions give rise to essentially the same MTCs as those obtained from quantum groups of types B, C and D at roots of unity. An advantage of this geometric approach is that the braid representations are more transparent than in the quantum group construction, although one pays for this convenience by having a less natural description of objects.

As we noted in the introduction, MTCs have also been constructed from representation categories of certain vertex operator algebras (VOAs) by Huang [**Hu**]. Rigidity and modularity are the most difficult to verify, while the monoidal structure was previously obtained. The allure of this approach is that it includes a proof of a very general form of the Verlinde conjecture from conformal field theory. Although this VOA construction of MTCs is more difficult than other approaches, it gives credence to the thesis that MTCs describe symmetries in quantum physical systems.

There are two indirect constructions that should be mentioned. One is the quantum double technique of Müger [**Mg**] (inspired by the double of a Hopf algebra) by which an MTC is constructed by "doubling" a monoidal category with some further technical properties. An example of this approach is the finite group algebra construction mentioned above. Bruguières [**Br**] describes conditions under which one may *modularize* a category that satisfies all of the axioms of an MTC except the invertibility of the S-matrix (called a **pre-modular category**). This corresponds essentially to taking a quotient or sub-category that does satisfy the modularity axiom.

4. Modularity, Unitarity and Ranks for Quantum Groups

There remains a fair amount of work to be done to have a complete theory of abstract unitary modular tensor categories; however, for quantum groups much is known. The condition of modularity has been settled for nearly all of the categories $\mathcal{C}(\mathfrak{g}, \ell, q)$, as well as the question of unitarity.

The modularity condition is often difficult to verify. Recently a **modularity criterion** was proved that sometimes simplifies the work (see [**Br**]):

Theorem 4.1 (Bruguìeres). *Suppose $\mathcal{O}$ is a pre-modular category, and let $\{X_0 = \mathbb{1}, X_1, \ldots, X_{n-1}\}$ be a set of representatives of the simple isomorphism classes. Then $\mathcal{O}$ is modular if and only if*

$$\mathcal{N} := \{X_i : S_{i,j} = d_i d_j \textit{ for all } X_j\} = \{\mathbb{1}\}.$$

X_r	A_r	B_r, r odd	D_r, r even	D_r, r odd	E_6	E_7
d	$r+1$	2	2	4	3	2

TABLE 1. Values of d for Lie algebra types with $d \neq 1$

Observe that one has $S_{0,j} = d_0 d_j = d_j$. The non-trivial elements of $\mathcal{N}$ are the **obstructions to modularity**, *i.e.* the objects for which the corresponding columns in the S-matrix are scalar multiples of the first column.

In the following subsections we describe the modularity and unitarity of the categories $\mathcal{C}(\mathfrak{g}, \ell, q)$, first for the cases that can be handled uniformly, and then for those that must be considered individually as well as a few subcategories of interest. Subsection 4.7 concerns the ranks of the categories $\mathcal{C}(\mathfrak{g}, \ell, q)$ and can be safely skipped by those readers not interested in this issue.

4.1. Uniform Cases $m \mid \ell$. For the categories $\mathcal{C}(\mathfrak{g}, \ell, q)$ the cases where ℓ is divisible by m have been mainly studied in the literature. The invertibility of S for Lie types A and C with $q = e^{\pi i/\ell}$ was shown in [**TW1**] using the work of Kac and Peterson [**KP**], and a complete treatment (for all Lie types with $q = e^{\pi i/\ell}$) is found in [**Ki**]. The invertibility can be extended to other values of q by the following Galois argument, which is found in [**TW2**] in a different form. By Formula (3.1) we see that the entries of the S-matrix:

$$S_{\lambda\mu} = (const.) \sum_{w \in W} \varepsilon(w) q^{2\langle w(\lambda+\rho), \mu+\rho\rangle}$$

are polynomials in $q^{1/d}$ where $d \in \mathbb{N}$ is minimal so that $d\langle\lambda, \mu\rangle \in \mathbb{Z}$ for all weights λ, μ. Thus $\det(S)$ is non-zero for any Galois conjugate of $e^{\pi i/d\ell}$, *i.e.* for any $q = e^{z\pi i/\ell}$ with $\gcd(z, d\ell) = 1$. Table 1 lists the values of d for all Lie types for which $d \neq 1$. Notice that there are sub-cases for types B and D. When $d = 1$ and $m|\ell$ the uniform case covers all possibilities, since then the condition $\gcd(z, d\ell) = 1$ is equivalent to the original assumption that q^2 is a primitive ℓth root of unity. So when $m|\ell$, the cases B_r with r even, C_r, E_8, F_4, and G_2 do not require further attention. If $m = 1$ and $\gcd(\ell, d) \neq 1$ the condition $\gcd(z, d\ell) = 1$ also degenerates to the original assumption that q^2 is a primitive ℓth root of unity so we need not consider D_r with ℓ even, E_6 with $3|\ell$ or E_7 with ℓ even.

Following a conjecture of Kirillov Jr. [**Ki**], Wenzl [**W**] showed that the Hermitian form on $\mathcal{C}(\mathfrak{g}, \ell, q)$ is positive definite for the uniform cases for certain values of q, and Xu [**X**] independently showed some of the cases covered by Wenzl. Their results are summarized in:

THEOREM 4.2 (Wenzl/Xu). *The categories $\mathcal{C}(\mathfrak{g}, \ell, q)$ are unitary when $m|\ell$ and $q = e^{\pi i/\ell}$.*

4.2. Type A. For Lie type A_r corresponding to $\mathfrak{g} = \mathfrak{sl}_{r+1}$ we have $m = 1$ and $d = r+1$. Bruguières [**Br**] shows that one has modularity for $q = e^{z\pi i/\ell}$ *if and only if* $\gcd(z, (r+1)\ell) = 1$. Moreover, Masbaum and Wenzl [**MW**] show that when $\gcd(\ell, r+1) = 1$ the subcategory of $\mathcal{C}(\mathfrak{sl}_{r+1}, \ell, q)$ generated by the simple objects labelled by integer weights is a modular subcategory whose rank is $1/(r+1)$ times the rank of the full category. There are a number of other proofs of this fact, see e.g. [**Br**] Section 5. Denote this subcategory by $\mathbb{Z}(A_r)$, and see Subsection 5.1.

4.3. Type B, ℓ odd. The category $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$ with ℓ odd has been considered to some extent by several authors including Sawin [**S1**], [**S2**] and Le-Turaev [**LT**]. It is shown in ([**TW2**], Theorem 9.9) that if ℓ is odd, the subcategory of $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$ generated by simple objects labelled by integer weights is modular and has rank exactly half of that of $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$. Combining the computations in [**R1**] and the modularity criterion of [**Br**] one has:

Theorem 4.3. *The category $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$ with ℓ odd is modular if and only if $q^\ell = -1$ and r is odd.*

Proof. By the modularity criterion we wish to show that there are obstructions to modularity (*i.e.* non-trivial objects in the set $\mathcal{N}$, see Theorem 4.1) if and only if the conditions of the theorem are not satisfied. By the modularity of the subcategory generated by simple objects labelled by integer weights, any obstructing object must be labelled by a half-integer weight. In [**R1**] the object X_γ labelled by the (half-integer) weight that is furthest from the 0 weight in the fundamental alcove is shown to induce an involution of the fundamental alcove (by tensoring with X_γ) that preserves q-dimension up to a sign. This implies that X_γ is the only potential obstruction to modularity. In [**R1**] (Scholium 4.11) the signs of the q-dimensions are analyzed, and the theorem then follows from the explicit computations of $N_{\gamma,\lambda}^{\nu}$, d_λ and θ_λ (also found in [**R1**]) together with Formula (2.2) and the obstruction equation $S_{\gamma,\lambda} = d_\gamma d_\lambda$. $\square$

The subject of the author's thesis [**R2**] (the results of which can be found in [**R1**]) is the question of unitarizability of the family of categories $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$ with ℓ odd. Using an analysis of the characters of the Grothendieck semirings it is shown that no member of this family of categories is unitary. In fact, there is a much stronger statement, for which we need the following definition:

Definition 4.4. A pre-modular category $\mathcal{O}$ is called **unitarizable** if $\mathcal{O}$ is tensor equivalent to a unitary pre-modular category $\mathcal{O}'$. By tensor equivalent we mean there exists a functor preserving the monoidal structure that is bijective on morphisms and such that every object in the target category is isomorphic to an object in the image of the functor.

Using a structure theorem of Tuba and Wenzl [**TbW**] it is shown in [**R2**] that:

Theorem 4.5. *Fix q with q^2 a primitive ℓth root of unity, ℓ odd, and r satisfying $2(2r+1) < \ell$. Then* no *braided tensor category having the same Grothendieck semiring as $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$ is unitarizable.*

Remark 4.6. When $\ell < 2(2r+1)$ the rank of $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$ is relatively small and the fusion rules of the category may coincide with those of another category that is known to be unitarizable. For example $\mathcal{C}(\mathfrak{so}_5, 7, q)$ has the same Grothendieck semiring as $\mathcal{C}(\mathfrak{sl}_2, 7, q)$ which is unitary for $q = e^{\pi i/7}$.

4.4. Type C, ℓ odd. For type C one has $m = 2$, so it remains to analyze the cases with ℓ odd. For this, we resort to the "rank-level duality" result of [**R1**] (Corollary 6.6) showing that the categories $\mathcal{C}(\mathfrak{so}_{2r+1}, \ell, q)$ and $\mathcal{C}(\mathfrak{sp}_{\ell-2r-1}, \ell, q)$ are tensor equivalent. Theorem 4.5 immediately implies these categories are not unitarizable for ℓ odd if $2(2r+1) < \ell$. Moreover, the technique in the proof of Theorem 4.3 can be applied to this case using the explicit values of d_λ and θ_λ and the image of the object X_γ under the tensor equivalence afforded by this rank-level duality. We then have:

THEOREM 4.7. *If ℓ is odd, the categories $\mathcal{C}(\mathfrak{sp}_{2r}, \ell, q)$ are not modular and if in addition $2(2r+1) < \ell$ they are not unitarizable.*

4.5. Remaining Types D, E_6 and E_7 Cases. The only remaining question for the sub-cases not covered by the uniform case is whether the condition $\gcd(z, d\ell) = 1$ is necessary for modularity. For Lie types D and E_7 the sub-cases correspond to ℓ odd, and for Lie type E_6 the sub-cases correspond to $3 \nmid \ell$. In our opinion this question is still open, of limited interest and one probably does not get modularity.

4.6. Types F_4 with ℓ odd, and G_2 with $3 \nmid \ell$. To our knowledge both the question of modularity and unitarizability are still open for F_4 with ℓ odd and G_2 with $3 \nmid \ell$. In light of the results in the Lie types B for ℓ odd (see Theorems 4.3 and 4.5), one might expect to find that these categories are not unitarizable (except possibly for small ℓ), but sometimes modular.

4.7. Generating Functions for $|C_\ell(\mathfrak{g})|$. For applications it is useful to know the ranks of the categories $\mathcal{C}(\mathfrak{g}, \ell, q)$.

We define an auxiliary label $\ell_m = 0$ if $m \mid \ell$ and $\ell_m = 1$ if $m \nmid \ell$ for notational convenience. We reduce the problem of determining the cardinalities of the labeling sets $C_\ell(\mathfrak{g})$ of simple objects to counting partitions of n with parts in a fixed (finite) multiset $\mathcal{S}(\mathfrak{g}, \ell_m)$ that depends only on the rank and Lie type of $\mathfrak{g}$ and the divisibility of ℓ by m. Fix a simple Lie algebra $\mathfrak{g}$ of rank r and a positive integer ℓ. Let $\lambda = \sum_i a_i \lambda_i$ be a dominant weight of $\mathfrak{g}$ written as an $\mathbb{N}$-linear combination of fundamental weights λ_i. To determine if $\lambda \in C_\ell(\mathfrak{g})$, we compute:

$$\langle \lambda + \rho, \vartheta_j \rangle = \langle \rho, \vartheta_j \rangle + \sum_i^r a_i \langle \lambda_i, \vartheta_j \rangle$$

where $j = 0$ or 1 depending on if $m \mid \ell$ or not. Setting $L_i^{(j)} = \langle \lambda_i, \vartheta_j \rangle$ we see that the condition that $\lambda \in C_\ell(\mathfrak{g})$ becomes:

$$\sum_i^k a_i L_i^{(j)} \leq \ell - \langle \rho, \vartheta_j \rangle - 1.$$

Since $a_i, L_i^{(j)} \in \mathbb{N}$ we have:

LEMMA 4.8. *The cardinality of $C_\ell(\mathfrak{g})$ is the number of partitions of all natural numbers n, $0 \leq n \leq \ell - \langle \rho, \vartheta_j \rangle - 1$ into parts from the size $r = \mathrm{rank}(\mathfrak{g})$ multiset $\mathcal{S}(\mathfrak{g}, \ell_m) = [L_i^{(j)}]_i^r$.*

So it remains only to compute the numbers $\langle \rho, \vartheta_j \rangle$ and $L_i^{(j)}$ (with $j = 0, 1$) for each Lie algebra $\mathfrak{g}$ and integer $\ell > \langle \rho, \vartheta_j \rangle$ and to apply standard combinatorics to count the number of partitions into parts in $\mathcal{S}(\mathfrak{g}, \ell_m)$. The first task is easily accomplished with the help of the book [**Bo**]. Table 2 lists the results of these computations, where $\ell_0 := \min\{\ell : \ell \geq \langle \rho, \vartheta_j \rangle + 1\}$ is the minimal non-degenerate value of ℓ.

Let $P_{\mathcal{T}}(n)$ denote the number of partitions of n into parts in a multiset $\mathcal{T}$, and $P_{\mathcal{T}}[s] = \sum_{n=0}^s P_{\mathcal{T}}(n)$ the number of partitions of all integers $0 \leq n \leq s$ into parts

TABLE 2. $\mathcal{C}(\mathfrak{g},q,\ell)$ Data

X_r	$\mathcal{S}(\mathfrak{g},\ell_m)$	ℓ_0
A_r	$[1,\dots,1]$	$r+1$
B_r, ℓ odd	$[1,2,\dots,2]$	$2r+1$
B_r, ℓ even	$[2,2,4,\dots,4]$	$4r-2$
C_r, ℓ odd	$[1,2,\dots,2]$	$2r+1$
C_r, ℓ even	$[2,\dots,2]$	$2r+2$
D_r	$[1,1,1,2,\dots,2]$	$2r-2$
E_6	$[1,1,2,2,2,3]$	12
E_7	$[1,2,2,2,3,3,4]$	18
E_8	$[2,2,3,3,4,4,5,6]$	30
F_4, ℓ even	$[2,4,4,6]$	18
F_4, ℓ odd	$[2,2,3,4]$	13
G_2, $3\mid\ell$	$[3,6]$	12
G_2, $3\nmid\ell$	$[2,3]$	7

from the multiset $\mathcal{T}$. Any standard reference on generating functions (see e.g. [**Sn**]) will provide enough details about generating functions to prove the following:

LEMMA 4.9. *The number $P_{\mathcal{T}}(n)$ of partitions of n into parts from the multiset $\mathcal{T}$ has generating function:*

$$\prod_{t\in\mathcal{T}}\frac{1}{1-x^t}=\sum_{n=0}^{\infty}P_{\mathcal{T}}(n)x^n,$$

while the number $P_{\mathcal{T}}[s]$ of partitions of all $n\in\mathbb{N}$ with $0\leq n\leq s$ into parts from the multiset $\mathcal{T}$ has generating function:

$$\frac{1}{1-x}\prod_{t\in\mathcal{T}}\frac{1}{1-x^t}=\sum_{s=0}^{\infty}P_{\mathcal{T}}[s]x^s.$$

Applying this lemma to the sets $\mathcal{S}(\mathfrak{g},\ell_m)$ given in Table 2 we obtain:

THEOREM 4.10. *Define*

$$F_{\mathfrak{g},\ell_m}(x)=\frac{1}{1-x}\prod_{k\in\mathcal{S}(\mathfrak{g},\ell_m)}\frac{1}{1-x^k}.$$

Then the rank $|C_\ell(\mathfrak{g})|$ of the pre-modular category $\mathcal{C}(\mathfrak{g},q,\ell)$ is the coefficient of

$$x^{\ell-\ell_0+\ell_m}$$

in the Taylor series expansion of $F_{\mathfrak{g},\ell_m}(x)$.

PROOF. It is clear from Lemma 4.9 that the coefficients of generating function $F_{\mathfrak{g},\ell_m}(x)$ counts the appropriate partitions. The coefficient of x that gives the rank for a specific ℓ is shifted by the minimal non-degenerate ℓ_0, which corresponds to the $x^0=1$ term if $m\mid\ell$ and to the $x^1=x$ term of $m\nmid\ell$, hence the correction by x^{ℓ_m}. With this normalization only the coefficients of those powers of x divisible (resp. indivisible) by m give ranks corresponding to ℓ divisible (resp. indivisible) by m. □

We illustrate the application of this theorem with some examples.

EXAMPLE 4.11. Let $\mathfrak{g}$ be of type G_2.

(a) Let $\ell = 27$. Then $\ell_m = 0$ and $\ell_0 = 12$. So the rank of $\mathcal{C}(\mathfrak{g}(G_2), q, 27)$ is given by the $(27-12+0) = 15$th coefficient of

$$\frac{1}{(1-x)(1-x^3)(1-x^6)} = (1+x+x^2)(1+2x^3+4x^6+6x^9+9x^{12}+12x^{15}+\cdots)$$

so $|C_{27}(\mathfrak{g}(G_2))| = 12$.

(b) Let $\ell = 14$. Then $\ell_m = 1$ and $\ell_0 = 7$. So $|C_{14}(\mathfrak{g}(G_2))|$ is the $(14-7+1)$th coefficient of

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = 1+x+2x^2+3x^3+4x^4+5x^5+7x^6+8x^7+10x^8\cdots$$

so the rank of $\mathcal{C}(\mathfrak{g}(G_2), q, 14)$ is 10.

5. Examples

We provide examples of two pre-modular categories, one of which is modular and unitary, while the other is not modular but has a (non-unitary) modular subcategory. We only give enough information to discuss the modularity and unitarity of the category.

5.1. Type $\mathbb{Z}(A_1)$ at $\ell = 5$. The following MTC is obtained from $\mathcal{C}(\mathfrak{sl}_2, 5, e^{\pi i/5})$ by taking the subcategory of modules with integer highest weights. There are two simple objects $\mathbb{1}$,and X_1 satisfying fusion rules: $X_1 \otimes X_1 = \mathbb{1} \oplus X_1$ and $\mathbb{1} \otimes X_i = X_i$. The S-matrix is $S = \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix}$ and the twists: $\theta_0 = 1$, $\theta_1 = e^{4\pi i/5}$. It is clear that $\det(S) \neq 0$, and it follows from [**W**] that the category is unitary (notice that the categorical dimensions are both positive).

5.2. Type B_2 at 9th Roots of Unity. Consider the pre-modular categories $\mathcal{C}(\mathfrak{so}_5, 9, e^{j\pi i/9})$ with $\gcd(18, j) = 1$. There are 12 inequivalent isomorphism classes of simple objects. The simple iso-classes of objects are labelled by $(\lambda_1, \lambda_2) \in \frac{1}{2}(\mathbb{N}^2)$ with $\lambda_1 \geq \lambda_2$. The twist coefficients for X_λ is $q^{\langle \lambda+2\rho, \lambda\rangle}$ where the form is twice the usual Euclidean form. The obstruction to modularity mentioned in the proof of Theorem 4.3 is labelled by $\gamma := \frac{1}{2}(5,5)$ The categorical dimension function is:

$$d_\lambda := \frac{[2(\lambda_1+\lambda_2+2)][2(\lambda_1-\lambda_2+1)][2\lambda_1+3][2\lambda_2+1]}{[4][3][2][1]}.$$

One checks that the simple object X_γ is indeed the cause of the singularity of the S-matrix, that is, $S_{\gamma,\lambda} = d_\gamma d_\lambda$ for all λ. Thus this category is not modular by Bruguières' criterion, Theorem 4.1.

Now let us consider the subcategory of $\mathcal{C}(\mathfrak{so}_5, 9, e^{j\pi i/9})$ with $\gcd(18, j) = 1$ generated by the simple objects labelled by integer weights:

$$\{(0,0),(1,0),(2,0),(1,1),(2,1),(2,2)\}.$$

The braiding and twists from the full category restrict, so the entries of the S-matrix are computed from Formula (2.2). Taking the ordering of simple objects above, we denote the categorical dimensions by d_i $0 \leq i \leq 5$. The fusion matrix corresponding to $(1,0)$ is:

$$N_1 := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is not hard to show that N_1 determines the other five fusion matrices by observing that N_1 has six distinct eigenvalues and the fusion matrices commute. There are a total of six categories corresponding to the six possible values of q. To describe the S-matrices we let α be a primitive 18th root of unity, and set $r_1 = -\alpha - \alpha^2 + \alpha^5$, $r_2 = \alpha + \alpha^2 - \alpha^4$ and $r_3 = \alpha^4 - \alpha^5$. Then we get the following S-matrices (for the 6 choices of α):

$$\begin{pmatrix} 1 & r_2 & r_3 & 1 & -1 & r_1 \\ r_2 & 1 & 1 & r_1 & -r_3 & 1 \\ r_3 & 1 & 1 & r_2 & -r_1 & 1 \\ 1 & r_1 & r_2 & 1 & -1 & r_3 \\ -1 & -r_3 & -r_1 & -1 & 1 & -r_2 \\ r_1 & 1 & 1 & r_3 & -r_2 & 1 \end{pmatrix}.$$

One checks that $\det(S) \neq 0$ for any α, so these categories *are* modular. A bit of Galois theory shows that there are only three distinct S for the six choices of α. Notice that it is already clear that the first column of S is never positive, since both 1 and -1 appear regardless of the choice of α. So none of these categories is unitary.

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