PERTURBATIONS OF SUBALGEBRAS OF TYPE II_1 FACTORS

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Abstract

In this paper we consider two von Neumann subalgebras $\mathcal{B}_0$ and $\mathcal{B}$ of a type II_1 factor $\mathcal{N}$. For a map ϕ on $\mathcal{N}$, we define

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2\colon \|x\| \le 1\},$$

and we measure the distance between $\mathcal{B}_0$ and $\mathcal{B}$ by the quantity $\|\mathbb{E}_{\mathcal{B}_0} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$. Under the hypothesis that the relative commutant in $\mathcal{N}$ of each algebra is equal to its center, we prove that close subalgebras have large compressions which are spatially isomorphic by a partial isometry close to 1 in the $\|\cdot\|_2$–norm. This hypothesis is satisfied, in particular, by masas and subfactors of trivial relative commutant. A general version with a slightly weaker conclusion is also proved. As a consequence, we show that if $\mathcal{A}$ is a masa and $u \in \mathcal{N}$ is a unitary such that $\mathcal{A}$ and $u\mathcal{A}u^*$ are close, then u must be close to a unitary which normalizes $\mathcal{A}$. These qualitative statements are given quantitative formulations in the paper.

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1 Introduction

In this paper we study pairs of von Neumann subalgebras $\mathcal{A}$ and $\mathcal{B}$ of a type II_1 factor $\mathcal{N}$ under the assumption that they are close to one another in a sense made precise below. Some of our results are very general, but the motivating examples are masas, subfactors, or algebras whose relative commutants in $\mathcal{N}$ equal their centers. In these special cases significant extra information is available beyond the general case. Ideally, two close subalgebras would be unitarily conjugate by a unitary close to the identity, but this is not true. In broard terms, we show that two close subalgebras can be cut by projections of large trace in such a way that the resulting algebras are spatially isomorphic by a partial isometry close to the identity. The exact nature of the projections and partial isometry depends on additional hypotheses placed on the subalgebras. Our results are an outgrowth of some recent work of the first author who proved a technical rigidity result for two masas $\mathcal{A}$ and $\mathcal{B}$ in a type II_1 factor $\mathcal{N}$ that has yielded several important results about type II_1 factors, [18, 19]. The techniques of these papers were first developed in [15]. This paper uses further refinements of these methods to prove the corresponding stability of certain subalgebras in separable type II_1 factors (Theorems 6.4 and 6.5). Several of the lemmas used below are modifications of those in [2, 18, 19], and versions of these lemmas go back to the the foundations of the subject in the papers of Murray, von Neumann, McDuff and Connes. Although the focus of this paper was initially the topic of masas, our results have been stated for general von Neumann algebras since the proofs are in a similar spirit. The crucial techniques from [18, 19] are the use of the pull down map $\Phi : L^1(\langle \mathcal{N}, \mathcal{B} \rangle) \to L^1(\mathcal{N})$, and detailed analyses of projections, partial isometries and module properties. The contractivity of Φ in the $\|\cdot\|_1$–norm, [19], is replaced here by a discussion of unbounded operators and related norm estimates in Lemma 5.1.

If $\phi\colon \mathcal{N} \to \mathcal{N}$ is a bounded linear map, then $\|\phi\|_{\infty,2}$ denotes the quantity

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2 \colon \ \|x\| \le 1\}, \tag{1.1}$$

and $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$ measures the distance between two subalgebras $\mathcal{A}$ and $\mathcal{B}$, where $\mathbb{E}_{\mathcal{A}}$ and $\mathbb{E}_{\mathcal{B}}$ are the associated trace preserving conditional expectations. We regard two subalgebras as close to one another if $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$ is small. A related notion is that of δ–containment, introduced in [11] and studied in [2]. We say that $\mathcal{A} \subset_\delta \mathcal{B}$ if, for each $a \in \mathcal{A}$, $\|a\| \le 1$, there exists $b \in \mathcal{B}$ such that $\|a - b\|_2 \le \delta$. This is equivalent to requiring that $\|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \leq \delta$, and so the condition $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq \delta$ implies δ–containment in both directions, so we will often use the norm inequality in the statement of results (see Remark 6.6).

A significant portion of the paper is devoted to the study of masas. Two metric based invariants have been introduced to measure the degree of singularity of a masa $\mathcal{A}$ in a type II_1 factor. In [13], the delta invariant $\delta(\mathcal{A})$ was introduced, taking values in $[0, 1]$. Motivated by this and certain examples arising from discrete groups, strong singularity and α–strong singularity for masas were defined and investigated in [22]. The singular masas are those which contain their groups of unitary normalizers, [5], and within this class the notion of a strongly singular masa $\mathcal{A} \subseteq \mathcal{N}$, [22], is defined by the inequality

$$\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2 \leq \|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \tag{1.2}$$

for all unitaries $u \in \mathcal{N}$. Such an inequality implies that each normalizing unitary lies in $\mathcal{A}$, so (1.2) can only hold for singular masas. We may weaken (1.2) by inserting a constant $\alpha \in (0, 1]$ on the left hand side, and a masa which satisfies the modified inequality is called α-strongly singular. Recently, [18], $\delta(\mathcal{A})$ was shown to be 1 for all singular masas. This result, [18, Cor. 2], may be stated as follows. If $\mathcal{A}$ is a singular masa in a type II_1 factor $\mathcal{N}$ and v is a partial isometry in $\mathcal{N}$ with vv^* and v^*v orthogonal projections in $\mathcal{A}$, then

$$\|vv^*\|_2 = \sup\{\|x - \mathbb{E}_{\mathcal{A}}(x)\|_2 : \ x \in v\mathcal{A}v^*, \ \|x\| \leq 1\}. \tag{1.3}$$

This result supports the possibility that all singular masas are strongly singular. Although we have not proved that singularity implies strong singularity, we have been able to establish the inequality

$$\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2 \leq 90\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \tag{1.4}$$

for all singular masas $\mathcal{A}$ in separably acting type II_1 factors $\mathcal{N}$. The constant 90 emerges from a chain of various estimates; our expectation is that it should be possible to replace it with a constant equal to or close to 1. The method of proof in [18] (and of the main technical lemma in [19]) uses the convexity techniques of Christensen, [2], together with the pull–down identity of Φ from [12], some work by Kadison on center–valued traces, [9], and fine estimates on projections. Our main proof (Theorem 5.2) follows that of [18], and also requires approximation of finite projections in $L^\infty[0, 1]\overline{\otimes}B(H)$. These are combined with a detailed handling of various inequalities involving projections and partial isometries.

There are two simple ways in which masas $\mathcal{A}$ and $\mathcal{B}$ in a type II_1 factor can be close in the $\|\cdot\|_{\infty,2}$–norm on their conditional expectations. If u is a unitary close to $\mathcal{A}$ in $\|\cdot\|_2$–norm and $u\mathcal{A}u^* = \mathcal{B}$, then $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$ is small. Secondly, if there is a projection q of large trace in $\mathcal{A}$ and $\mathcal{B}$ with $q\mathcal{A} = q\mathcal{B}$, then again $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$ is small. In Theorem 6.5 we show that a combination of these two methods is the only way in which $\mathcal{A}$ can be close to $\mathcal{B}$ in separably acting factors.

The structure of the paper is as follows. Section 2 contains preliminary lemmas which include statements of some known results that will be used subsequently. The operator h in Proposition 2.4 was important for Christensen's work in [2] and plays an essential role here. Theorem 2.6 investigates its spectrum to aid in later estimates. The third section deals with two close algebras, one of which is contained in the other. We present a sequence of lemmas, the purpose of which is to cut the algebras by a large projection so that equality results. The fourth section collects some more background results preparatory to the next section where it is shown that two close subalgebras can be cut so that they become isomorphic by a suitably chosen partial isometry. In the final section we focus attention on applying these results to masas. One consequence is that if a unitary conjugate $u\mathcal{A}u^*$ of a masa $\mathcal{A}$ is close to the original masa then u must be close to a normalizing unitary, and this allows us to present the results on strongly singular masas mentioned above.

The crucial estimates are contained in Theorems 5.2, 3.5 and Corollary 2.5. We recommend reading these three results in the order stated, referring back to ancillary lemmas and propositions as needed. Corollary 2.5 is essentially due to Christensen in his pioneering paper [2], but without the norm inequalities which we have included. Two of our main results, Theorems 3.5 and 5.2, generalize [19, A.2] and use methods from [15, Section 4].

Our results are formulated for subalgebras of a finite factor $\mathcal{N}$. In only a few places is this requirement necessary, Theorem 3.7 for example, and when the statement of a result makes sense for a von Neumann algebra $\mathcal{N}$ with a unital faithful normal trace, the same proof is valid.

2 Preliminaries

Let $\mathcal{N}$ be a fixed but arbitrary separably acting type II_1 factor with faithful normalized normal trace τ, and let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{N}$. The trace induces an inner product

$$\langle x, y\rangle = \tau(y^*x), \qquad x, y \in \mathcal{N}, \tag{2.1}$$

on $\mathcal{N}$. Then $L^2(\mathcal{N}, \tau)$ is the resulting completion with norm

$$\|x\|_2 = (\tau(x^*x))^{1/2}, \qquad x \in \mathcal{N}, \tag{2.2}$$

and when $x \in \mathcal{N}$ is viewed as a vector in this Hilbert space we will denote it by $\hat{x}$. Several traces will be used in the paper and so we will write $\|\cdot\|_{2,\mathrm{tr}}$ when there is possible ambiguity. The unique trace preserving conditional expectation $\mathbb{E}_{\mathcal{B}}$ of $\mathcal{N}$ onto $\mathcal{B}$ may be regarded as a projection in $B(L^2(\mathcal{N}), \tau)$, where we denote it by $e_{\mathcal{B}}$. Thus

$$e_{\mathcal{B}}(\hat{x}) = \widehat{\mathbb{E}_{\mathcal{B}}(x)}, \qquad x \in \mathcal{N}. \tag{2.3}$$

Properties of the trace show that there is a conjugate linear isometry $J\colon\ L^2(\mathcal{N}, \tau) \to L^2(\mathcal{N}, \tau)$ defined by

$$J(\hat{x}) = \widehat{x^*}, \qquad x \in \mathcal{N}, \tag{2.4}$$

and it is standard that $\mathcal{N}$, viewed as an algebra of left multiplication operators on $L^2(\mathcal{N}, \tau)$, has commutant $J\mathcal{N}J$. The von Neumann algebra generated by $\mathcal{N}$ and $e_{\mathcal{B}}$ is denoted by $\langle \mathcal{N}, \mathcal{B}\rangle$, and has commutant $J\mathcal{B}J$. If $\mathcal{B}$ is a maximal abelian self–adjoint subalgebra (masa) of $\mathcal{N}$, then $\langle \mathcal{N}, \mathcal{B}\rangle$ is a type I_∞ von Neumann algebra, since its commutant is abelian. Moreover, its center is $J\mathcal{B}J$, a masa in $\mathcal{N}'$ and thus isomorphic to $L^\infty[0,1]$. The general theory of type I von Neumann algebras, [10], shows that there is then a separable infinite dimensional Hilbert space H so that $\langle \mathcal{N}, \mathcal{B}\rangle$ and $L^\infty[0,1]\overline{\otimes}B(H)$ are isomorphic. When appropriate, for $\mathcal{B}$ a masa, we will regard an element $x \in \langle \mathcal{N}, \mathcal{B}\rangle$ as a uniformly bounded measurable $B(H)$–valued function $x(t)$ on [0,1]. Under this identification, the center $J\mathcal{B}J$ of $\langle \mathcal{N}, \mathcal{B}\rangle$ corresponds to those functions taking values in $\mathbb{C}I$. We denote by tr the unique semi-finite faithful normal trace on $B(H)$ which assigns the value 1 to each rank 1 projection.

The following lemma (see [12, 17]) summarizes some of the basic properties of $\langle \mathcal{N}, \mathcal{B}\rangle$ and $e_{\mathcal{B}}$.

Lemma 2.1. *Let $\mathcal{N}$ be a separably acting type* II_1 *factor with a von Neumann subalgebra $\mathcal{B}$. Then*

$$\text{(i)}\quad e_{\mathcal{B}} x e_{\mathcal{B}} = e_{\mathcal{B}} \mathbb{E}_{\mathcal{B}}(x) = \mathbb{E}_{\mathcal{B}}(x) e_{\mathcal{B}}, \qquad x \in \mathcal{N}; \tag{2.5}$$

$$\text{(ii)}\quad e_{\mathcal{B}} \langle \mathcal{N}, \mathcal{B} \rangle = \overline{e_{\mathcal{B}} \mathcal{N}}^{\,w}, \quad \langle \mathcal{N}, \mathcal{B} \rangle e_{\mathcal{B}} = \overline{\mathcal{N} e_{\mathcal{B}}}^{\,w}; \tag{2.6}$$

$$\text{(iii)}\quad \textit{if } x \in \mathcal{N} \cup J\mathcal{B}J \textit{ and } e_{\mathcal{B}} x = 0, \textit{ then } x = 0; \tag{2.7}$$

$$\text{(iv)}\quad e_{\mathcal{B}} \langle \mathcal{N}, \mathcal{B} \rangle e_{\mathcal{B}} = \mathcal{B} e_{\mathcal{B}} = e_{\mathcal{B}} \mathcal{B}; \tag{2.8}$$

(v) *there is a faithful normal semi-finite trace* Tr *on $\langle \mathcal{N}, \mathcal{B} \rangle$ which satisfies*

$$\mathrm{Tr}(x e_{\mathcal{B}} y) = \tau(xy), \qquad x, y \in \mathcal{N}, \tag{2.9}$$

and in particular,

$$\mathrm{Tr}(e_{\mathcal{B}}) = 1. \tag{2.10}$$

The following result will be needed subsequently. We denote by $\|x\|_{2,\mathrm{Tr}}$ the Hilbert space norm induced by Tr on the subspace of $\langle \mathcal{N}, \mathcal{B} \rangle$ consisting of elements satisfying $\mathrm{Tr}(x^*x) < \infty$.

Lemma 2.2. *Let $\mathcal{B}$ be a masa in $\mathcal{N}$ and let $\varepsilon > 0$. If $f \in \langle \mathcal{N}, \mathcal{B} \rangle$ is a projection of finite trace and*

$$\|f - e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \leq \varepsilon, \tag{2.11}$$

then there exists a central projection $z \in \langle \mathcal{N}, \mathcal{B} \rangle$ such that zf and $ze_{\mathcal{B}}$ are equivalent projections in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Moreover, the following inequalities hold:

$$\|zf - ze_{\mathcal{B}}\|_{2,\mathrm{Tr}}, \quad \|ze_{\mathcal{B}} - e_{\mathcal{B}}\|_{2,\mathrm{Tr}}, \quad \|zf - e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \leq \varepsilon. \tag{2.12}$$

Proof. From (2.8), $e_{\mathcal{B}}$ is an abelian projection in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ so, altering $e_{\mathcal{B}}(t)$ on a null set if necessary, each $e_{\mathcal{B}}(t)$ is a projection in $B(H)$ whose rank is at most 1. If $\{t \in [0,1]\colon\ e_{\mathcal{B}}(t) = 0\}$ were not a null set, then there would exist a non–zero central projection $p \in J\mathcal{B}J$ corresponding to this set so that $e_{\mathcal{B}} p = 0$, contradicting (2.7). Thus we may assume that each $e_{\mathcal{B}}(t)$ has rank 1.

Since Tr is a faithful normal semi-finite trace on $\langle \mathcal{N}, \mathcal{B} \rangle$, there exists a non-negative $\mathbb{R}$–valued measurable function $k(t)$ on [0,1] such that

$$\mathrm{Tr}(y) = \int_0^1 k(t)\mathrm{tr}(y(t))dt, \quad y \in \langle \mathcal{N}, \mathcal{B} \rangle, \quad \mathrm{Tr}(y^*y) < \infty. \tag{2.13}$$

By (2.10),

$$\mathrm{Tr}(e_{\mathcal{B}}) = \int_0^1 k(t)\mathrm{tr}(e_{\mathcal{B}}(t))dt = 1, \tag{2.14}$$

and thus integration against $k(t)$ defines a probability measure μ on [0,1] such that

$$\mathrm{Tr}(y) = \int_0^1 \mathrm{tr}(y(t))d\mu(t), \quad y \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle, \quad \mathrm{Tr}(y^*y) < \infty. \tag{2.15}$$

It follows from (2.15) that

$$\|y\|_{2,\mathrm{Tr}}^2 = \int_0^1 \mathrm{tr}(y(t)^*y(t))d\mu(t), \qquad y \in \langle \mathcal{N}, \mathcal{B} \rangle. \tag{2.16}$$

Consider a rank 1 projection $p \in B(H)$ and a projection $q \in B(H)$ of rank $n \geq 2$. Then

$$\mathrm{tr}((p-q)^2) = \mathrm{tr}(p+q-2pq) = \mathrm{tr}(p+q-2pqp) \geq \mathrm{tr}(p+q-2p) \geq 1, \tag{2.17}$$

and the same inequality is obvious if $q = 0$. Let $G = \{t \in [0,1]\colon\ \mathrm{rank}(f(t)) \neq 1\}$. Then, from (2.11),

$$\varepsilon^2 \geq \|f - e_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2 \geq \int_G \mathrm{tr}((f(t) - e_{\mathcal{B}}(t))^2)d\mu(t) \geq \mu(G), \tag{2.18}$$

by (2.17). Let $z = \chi_{G^c} \otimes I$, a central projection in $\langle \mathcal{N}, \mathcal{B} \rangle$. Then the ranks of $z(t)f(t)$ and $z(t)e_{\mathcal{B}}(t)$ are simultaneously 0 or 1, and so zf and $ze_{\mathcal{B}}$ are equivalent projections in $\langle \mathcal{N}, \mathcal{B} \rangle$. Then

$$\|ze_{\mathcal{B}} - e_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2 = \int_G \mathrm{tr}(e_{\mathcal{B}}(t))d\mu(t) = \mu(G) \leq \varepsilon^2, \tag{2.19}$$

from (2.18), while

$$\begin{aligned} \|zf - e_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2 &= \int_G \mathrm{tr}(e_{\mathcal{B}}(t))d\mu(t) + \int_{G^c} \mathrm{tr}((f(t) - e_{\mathcal{B}}(t))^2)d\mu(t) \\ &\leq \|f - e_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2 \end{aligned} \tag{2.20}$$

since, on G,

$$\mathrm{tr}(e_{\mathcal{B}}(t)) = 1 \leq \mathrm{tr}((f(t) - e_{\mathcal{B}}(t))^2), \tag{2.21}$$

by (2.17). Finally,

$$\|zf - ze_{\mathcal{B}}\|_{2,\mathrm{Tr}} \leq \|z\|\|f - e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \leq \varepsilon, \tag{2.22}$$

completely the proof of (2.12). □

We now recall some properties of the polar decomposition and some trace norm inequalities. These may be found in [3, 10].

Lemma 2.3. *Let $\mathcal{M}$ be a von Neumann algebra.*

(i) *If $w \in \mathcal{M}$ then there exists a partial isometry $v \in \mathcal{M}$, whose initial and final spaces are respectively the closures of the ranges of w^* and w, satisfying*

$$w = v(w^*w)^{1/2} = (ww^*)^{1/2}v. \tag{2.23}$$

(ii) *Suppose that $\mathcal{M}$ has a faithful normal semifinite trace* Tr*. If $x \in \mathcal{M}$, $0 \le x \le 1$, $\mathrm{Tr}(x^*x) < \infty$, and f is the spectral projection of x corresponding to the interval $[1/2,\, 1]$, then*

$$\|e - f\|_{2,\mathrm{Tr}} \le 2\|e - x\|_{2,\mathrm{Tr}} \tag{2.24}$$

for any projection $e \in \mathcal{M}$ of finite trace.

(iii) *Suppose that $\mathcal{M}$ has a faithful normal semifinite trace and let p and q be equivalent finite projections in $\mathcal{M}$. Then there exists a partial isometry $v \in \mathcal{M}$ and a unitary $u \in \mathcal{M}$ satisfying*

$$v^*v = p, \quad vv^* = q, \tag{2.25}$$

$$v|p-q| = |p-q|v, \tag{2.26}$$

$$|v-p|, |v-q| \le 2^{1/2}|p-q|, \tag{2.27}$$

$$upu^* = q, \quad u|p-q| = |p-q|u, \tag{2.28}$$

$$|1-u| \le 2^{1/2}|p-q|. \tag{2.29}$$

(iv) *Suppose that $\mathcal{M}$ has a faithful normal semifinite trace and let p and q be finite projections in $\mathcal{M}$. Then the partial isometry v in the polar decomposition of pq satisfies*

$$\|p - v\|_{2,\mathrm{Tr}},\ \|q - v\|_{2,\mathrm{Tr}} \le \sqrt{2}\|p - q\|_{2,\mathrm{Tr}}. \tag{2.30}$$

The following result is essentially in [2], and is also used in [18, 19]. We reprove it here since the norm estimates that we obtain will be crucial for subsequent developments. The operator h below will be important at several points and we will refer below to the procedure for obtaining it as *averaging $e_{\mathcal{B}}$ over $\mathcal{A}$*.

Proposition 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann subalgebras in a separably acting type* II_1 *factor $\mathcal{N}$, and let $\overline{K}^{w}_{\mathcal{A}}(e_{\mathcal{B}})$ be the weak closure of the set*

$$K_{\mathcal{A}}(e_{\mathcal{B}}) = \mathrm{conv}\,\{ue_{\mathcal{B}}u^*\colon\ u \text{ is a unitary in } \mathcal{A}\} \tag{2.31}$$

in $\langle \mathcal{N}, \mathcal{B} \rangle$*. Then* $\overline{K}^w_{\mathcal{A}}(e_{\mathcal{B}})$ *contains a unique element* h *of minimal* $\|\cdot\|_{2,\mathrm{Tr}}$*-norm, and this element satisfies*

$$\text{(i)} \quad h \in \mathcal{A}' \cap \langle \mathcal{N}, \mathcal{B} \rangle, \qquad 0 \le h \le 1; \tag{2.32}$$

$$\text{(ii)} \quad 1 - \mathrm{Tr}(e_{\mathcal{B}} h) \le \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|^2_{\infty,2}; \tag{2.33}$$

$$\text{(iii)} \quad \mathrm{Tr}(e_{\mathcal{B}} h) = \mathrm{Tr}(h^2); \tag{2.34}$$

$$\text{(iv)} \quad \|h - e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \le \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}. \tag{2.35}$$

Proof. Each $x \in K_{\mathcal{A}}(e_{\mathcal{B}})$ satisfies $0 \le x \le 1$, and so the same is true for elements of $\overline{K}^w_{\mathcal{A}}(e_{\mathcal{B}})$. Moreover, each $x \in K_{\mathcal{A}}(e_{\mathcal{B}})$ has unit trace by Lemma 2.1 (v). Let P be the set of finite trace projections in $\langle \mathcal{N}, \mathcal{B} \rangle$. Then, for $x \in \langle \mathcal{N}, \mathcal{B} \rangle$, $x \ge 0$,

$$\mathrm{Tr}(x) = \sup\{\mathrm{Tr}(xp) \colon \ p \in P\}. \tag{2.36}$$

If $p \in P$ and (x_α) is a net in $K_{\mathcal{A}}(e_{\mathcal{B}})$ converging weakly to $x \in \overline{K}^w_{\mathcal{A}}(e_{\mathcal{B}})$, then

$$\lim_\alpha \mathrm{Tr}(x_\alpha p) = \mathrm{Tr}(xp), \tag{2.37}$$

and it follows from (2.36) that $\mathrm{Tr}(x) \le 1$. Since $x^2 \le x$, it follows that $x \in L^2(\langle \mathcal{N}, \mathcal{B} \rangle, \mathrm{Tr})$. Since $\mathrm{span}\{P\}$ is norm dense in $L^2(\langle \mathcal{N}, \mathcal{B} \rangle, \mathrm{Tr})$, we conclude from (2.37) that (x_α) converges weakly to x in the Hilbert space. Thus $\overline{K}^w_{\mathcal{A}}(e_{\mathcal{B}})$ is weakly compact in both $\langle \mathcal{N}, \mathcal{B} \rangle$ and $L^2(\langle \mathcal{N}, \mathcal{B} \rangle, \mathrm{Tr})$, and so norm closed in the latter. Thus there is a unique element $h \in \overline{K}^w_{\mathcal{A}}(e_{\mathcal{B}})$ of minimal $\|\cdot\|_{2,\mathrm{Tr}}$ – norm.

For each unitary $u \in \mathcal{A}$, the map $x \mapsto uxu^*$ is a $\|\cdot\|_{2,\mathrm{Tr}}$ – norm isometry which leaves $\overline{K}^w_{\mathcal{A}}(e_{\mathcal{B}})$ invariant. Thus

$$uhu^* = h, \quad u \text{ unitary in } \mathcal{A}, \tag{2.38}$$

by minimality of h, so $h \in \mathcal{A}' \cap \langle \mathcal{N}, \mathcal{B} \rangle$. This proves (i).

Consider a unitary $u \in \mathcal{A}$. Then, by Lemma 2.1,

$$\begin{aligned} 1 - \mathrm{Tr}(e_{\mathcal{B}} u e_{\mathcal{B}} u^*) &= 1 - \mathrm{Tr}(e_{\mathcal{B}} \mathbb{E}_{\mathcal{B}}(u) u^*) = 1 - \tau(\mathbb{E}_{\mathcal{B}}(u) u^*) \\ &= 1 - \tau(\mathbb{E}_{\mathcal{B}}(u) \mathbb{E}_{\mathcal{B}}(u)^*) = 1 - \|\mathbb{E}_{\mathcal{B}}(u)\|_2^2 \\ &= \|(I - \mathbb{E}_{\mathcal{B}})(u)\|_2^2 \le \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|^2_{\infty,2}. \end{aligned} \tag{2.39}$$

This inequality persists when $ue_{\mathcal{B}}u^*$ is replaced by elements of $K_{\mathcal{A}}(e_{\mathcal{B}})$, so it follows from (2.37) that

$$1 - \mathrm{Tr}(e_{\mathcal{B}} h) \le \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|^2_{\infty,2}, \tag{2.40}$$

proving (ii).

Since $h \in \mathcal{A}' \cap \langle \mathcal{N}, \mathcal{B} \rangle$,

$$\mathrm{Tr}(ue_{\mathcal{B}}u^*h) = \mathrm{Tr}(e_{\mathcal{B}}u^*hu) = \mathrm{Tr}(e_{\mathcal{B}}h) \tag{2.41}$$

for all unitaries $u \in \mathcal{A}$. Part (iii) follows from this by taking suitable convex combinations and a weak limit to replace $ue_{\mathcal{B}}u^*$ by h on the left hand side of (2.41). Finally, using (2.40) and (2.41),

$$\begin{aligned}\|h - e_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2 &= \mathrm{Tr}(h^2 - 2he_{\mathcal{B}} + e_{\mathcal{B}}) \\ &= \mathrm{Tr}(e_{\mathcal{B}} - he_{\mathcal{B}}) \\ &= 1 - \mathrm{Tr}(he_{\mathcal{B}}) \\ &\leq \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}^2 \end{aligned} \tag{2.42}$$

proving (iv). □

For the last two results of this section, h is the element constructed in the previous proposition.

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann subalgebras of $\mathcal{N}$ and let f be the spectral projection of h corresponding to the interval*
$[1/2, 1]$. *Then $f \in \mathcal{A}' \cap \langle \mathcal{N}, \mathcal{B} \rangle$, and*

$$\|e_{\mathcal{B}} - f\|_{2,\mathrm{Tr}} \leq 2\|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}. \tag{2.43}$$

Proof. The first assertion is a consequence of elementary spectral theory. The second follows from Proposition 2.4 (iv) and Lemma 2.3 (ii). □

Theorem 2.6. *Let $\mathcal{Q}_0 \subseteq \mathcal{Q}_1$ be a containment of finite von Neumann algebras and let τ be a unital faithful normal trace on $\mathcal{Q}_1$. Suppose that*

$$\mathcal{Q}_0' \cap \mathcal{Q}_1 = \mathcal{Z}(\mathcal{Q}_0) = \mathcal{Z}(\mathcal{Q}_1), \tag{2.44}$$

and let $h \in \langle \mathcal{Q}_1, \mathcal{Q}_0 \rangle$ be the operator obtained from averaging $e_{\mathcal{Q}_0}$ over $\mathcal{Q}_1$. Then

$$h \in \mathcal{Z}(\mathcal{Q}_0) = \mathcal{Q}_1' \cap \langle \mathcal{Q}_1, \mathcal{Q}_0 \rangle = \mathcal{Z}(\mathcal{Q}_1), \tag{2.45}$$

and the spectrum of h lies in the set

$$S = \{(4\cos^2(\pi/n))^{-1} : n \geq 3\} \cup [0, 1/4]. \tag{2.46}$$

In particular, the spectrum of h lies in $\{1\} \cup [0, 1/2]$, and the spectral projection q_1 corresponding to $\{1\}$ is the largest central projection for which $\mathcal{Q}_0 q_1 = \mathcal{Q}_1 q_1$.

Proof. Let $\mathcal{Q}_2$ denote $\langle \mathcal{Q}_1, \mathcal{Q}_0 \rangle$. Then $\mathcal{Q}_2' = J\mathcal{Q}_0 J$, so $\mathcal{Q}_1' \cap \mathcal{Q}_2 = J\mathcal{Q}_1 J \cap (J\mathcal{Q}_0 J)'$, which is $J(\mathcal{Q}_0' \cap \mathcal{Q}_1)J = J(\mathcal{Z}(\mathcal{Q}_1))J = \mathcal{Z}(\mathcal{Q}_1)$. In addition, $\mathcal{Z}(\mathcal{Q}_2) = \mathcal{Z}(\mathcal{Q}_2') = \mathcal{Z}(J\mathcal{Q}_0 J) = Z(\mathcal{Q}_1)$, so the algebras $\mathcal{Z}(\mathcal{Q}_0)$, $\mathcal{Z}(\mathcal{Q}_1)$, $\mathcal{Z}(\mathcal{Q}_2)$, $\mathcal{Q}_0' \cap \mathcal{Q}_1$ and $\mathcal{Q}_1' \cap \mathcal{Q}_2$ coincide under these hypotheses. Since $h \in \mathcal{Q}_1' \cap \mathcal{Q}_2$, by Proposition 2.4, we have thus established (2.45).

The set S consists of an interval and a decreasing sequence of points. If the spectrum of h is not contained in S, then we may find a closed interval $[a, b] \subseteq S^c \cap (1/4, 1)$ so that the corresponding spectral projection z of h is non–zero and also lies in $\mathcal{Z}(\mathcal{Q}_1)$. By cutting the algebras by z, we may assume that $a1 \le h \le b1$ and $[a, b] \cap S = \emptyset$ so $a \ge 1/4$. The trace Tr on $\mathcal{Q}_2$ coming from the basic construction satisfies $\mathrm{Tr}(1) \le a^{-1}\mathrm{Tr}(h) = a^{-1}\mathrm{Tr}(e_{\mathcal{Q}_0}) \le 4$ and is thus finite. Let Ctr denote the center–valued trace on $\mathcal{Q}_2$, whose restrictions to $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are also the center–valued traces on these subalgebras. Then $\mathrm{Ctr}(e_{\mathcal{Q}_0}) \ge a1$. If $\mathcal{Q}_2$ has a central summand of type I_n with corresponding central projection p then cutting by p gives containment of type I_n algebras with equal centers and are thus equal to each other. This would show that $hp = p$ and 1 would lie in the spectrum of h, contrary to assumption. Thus $\mathcal{Q}_2$ is a type II_1 von Neumann algebra. Then 1 may be expressed as a sum of four equivalent projections $\{p_i\}_{i=1}^4$ each having central trace $(4^{-1})1$. Thus each p_i is equivalent to a subprojection of $e_{\mathcal{Q}_0}$ and so there exist partial isometries $v_i \in \mathcal{Q}_2$ such that $1 = \sum_{i=1}^4 v_i e_{\mathcal{Q}_0} v_i^*$. Since $e_{\mathcal{Q}_0}\mathcal{Q}_1 = e_{\mathcal{Q}_0}\mathcal{Q}_2$, we may replace each v_i by an operator $w_i \in \mathcal{Q}_1$, yielding $1 = \sum_{i=1}^4 w_i e_{\mathcal{Q}_0} w_i^*$. For each $x \in \mathcal{Q}_1$, multiply on the right by $x e_{\mathcal{Q}_0}$ to obtain

$$x e_{\mathcal{Q}_0} = \sum_{i=1}^{4} w_i \mathbb{E}_{\mathcal{Q}_0}(w_i^* x) e_{\mathcal{Q}_0}, \quad x \in \mathcal{Q}_1, \tag{2.47}$$

so $x = \sum_{i=1}^4 w_i \mathbb{E}_{\mathcal{Q}_0}(w_i^* x)$, and $\mathcal{Q}_1$ is a finitely generated right $\mathcal{Q}_0$–module. In a similar fashion $\mathcal{Q}_2 = \sum_{i=1}^4 w_i e_{\mathcal{Q}_0} \mathcal{Q}_2 = \sum_{i=1}^4 w_i e_{\mathcal{Q}_0} \mathcal{Q}_1$, and so $\mathcal{Q}_2$ is finitely generated over $\mathcal{Q}_1$. This is a standard argument in subfactor theory (see [12]) which we include for the reader's convenience.

Let Ω be the spectrum of $\mathcal{Z}(\mathcal{Q}_2)$, and fix $\omega \in \Omega$. Then

$$\mathcal{I}_2 = \{x \in \mathcal{Q}_2 \colon \mathrm{Ctr}(x^* x)(\omega) = 0\} \tag{2.48}$$

is a maximal norm closed ideal in $\mathcal{Q}_2$ and $\mathcal{Q}_2/\mathcal{I}_2$ is a type II_1 factor, denoted $\mathcal{M}_2$, with trace $\tau_\omega = \omega \circ \mathrm{Ctr}$, [21]. Similar constructions yield maximal ideals $\mathcal{I}_k \subseteq \mathcal{Q}_k$, $k = 0, 1$, and factors $\mathcal{M}_k = \mathcal{Q}_k/\mathcal{I}_k$. Equality of the centers gives $\mathcal{Q}_k \cap \mathcal{I}_2 = \mathcal{I}_k$ for $k = 0, 1$, and so $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2$ is an inclusion of factors. Let $\pi\colon \mathcal{Q}_2 \to \mathcal{M}_2$ denote the quotient map, and let $e = \pi(e_{\mathcal{Q}_0})$. From above we note that $\mathcal{M}_2$ is a finitely generated $\mathcal{M}_1$–module.

Consider $x \in \mathcal{I}_1$. By uniqueness of the center–valued trace, the composition of $\mathbb{E}_{\mathcal{Q}_0}$ with the restriction of Ctr to $\mathcal{Q}_0$ is Ctr. Thus

$$\mathrm{Ctr}(\mathbb{E}_{\mathcal{Q}_0}(x^*x)))(\omega) = \mathrm{Ctr}(x^*x)(\omega), \quad x \in \mathcal{Q}_1. \tag{2.49}$$

Conditional expectations are completely positive unital maps and so $\mathbb{E}_{\mathcal{Q}_0}(x^*)\mathbb{E}_{\mathcal{Q}_0}(x) \leq \mathbb{E}_{\mathcal{Q}_0}(x^*x)$, showing that $\mathbb{E}_{\mathcal{Q}_0}$ maps $\mathcal{I}_1$ to $\mathcal{I}_0$. Thus there is a well defined τ_ω–preserving conditional expectation $\mathbb{E}\colon \mathcal{M}_1 \to \mathcal{M}_0$ given by $\mathbb{E}(x + \mathcal{I}_1) = \mathbb{E}_{\mathcal{Q}_0}(x) + \mathcal{I}_0$ for $x \in \mathcal{Q}_1$. From above, e commutes with $\mathcal{M}_0$ and $\mathcal{M}_2$ is generated by $\mathcal{M}_1$ and e. Moreover, $exe = \mathbb{E}(x)e$ for $x \in \mathcal{M}_1$ by applying π to the equation $e_{\mathcal{Q}_0}xe_{\mathcal{Q}_0} = \mathbb{E}_{\mathcal{Q}_0}(x)e_{\mathcal{Q}_0}$ for $x \in \mathcal{Q}_1$. Thus $\mathcal{M}_2$ is the extension of $\mathcal{M}_1$ by $\mathcal{M}_0$ with Jones projection e, [7]. Now $\mathrm{Ctr}(e_{\mathcal{Q}_0})(\omega) = h(\omega) \in [a, b]$, so $\tau_\omega(e) = h(\omega)$ while $\tau_\omega(1) = 1$. It follows that $[\mathcal{M}_1\colon \mathcal{M}_0]^{-1} \in [a, b]$, contradicting the theorem of Jones, [7], on the possible values of the index.

Now let q_1 be the spectral projection of h corresponding to $\{1\}$. If we cut by q_1 then we may assume that $h = 1$. But then $e_{\mathcal{Q}_0} = 1$ and $\mathcal{Q}_0 = \mathcal{Q}_1$. We conclude that $\mathcal{Q}_0q_1 = \mathcal{Q}_1q_1$. On the other hand, let $z \in \mathcal{Z}(\mathcal{Q}_1)$ be a projection such that $\mathcal{Q}_0z = \mathcal{Q}_1z$. Then $e_{\mathcal{Q}_0}z = z$, so $hz = z$, showing that $z \leq q_1$. Thus q_1 is the largest central projection with the stated property. $\square$

3 Containment of finite algebras

In this section we consider an inclusion $\mathcal{M} \subseteq \mathcal{N}$ of finite von Neumann algebras where $\mathcal{N}$ has a faithful normal unital trace τ, and where $\mathcal{N} \subset_\delta \mathcal{M}$ for some small positive number δ. Our objective is to show that, by cutting the algebras by a suitable projection p in the center of the relative commutant $\mathcal{M}' \cap \mathcal{N}$ of large trace dependent on δ, we may arrive at $\mathcal{M}p = p\mathcal{N}p$. This is achieved in the following lemmas which are independent of one another. However, we have chosen the notation so that they may be applied sequentially to our original inclusion $\mathcal{M} \subseteq \mathcal{N}$. The definition of $\mathcal{N} \subset_\delta \mathcal{M}$ depends implicitly on the trace τ assigned to $\mathcal{N}$. Since we will be rescaling traces at various points, we will make this explicit by adopting the notation $\mathcal{N} \subset_{\delta,\tau} \mathcal{M}$. If $\tau_1 = \lambda\tau$ for some $\lambda > 0$, then

$$\|x\|_{2,\tau_1} = \sqrt{\lambda}\|x\|_{2,\tau}, \qquad x \in \mathcal{N}. \tag{3.1}$$

Consequently $\mathcal{N} \subset_{\delta,\tau} \mathcal{M}$ becomes $\mathcal{N} \subset_{\sqrt{\lambda}\ \delta,\tau_1} \mathcal{M}$ for this change of trace.

It is worth noting that the beginning of the proof of the next lemma shows that if $\mathcal{M}_0 \subseteq \mathcal{N}_0$, $\mathcal{N}_0 \subset_{\delta_0,\tau_0} \mathcal{M}_0$ for a von Neumann algebra $\mathcal{N}_0$ with a faithful normal unital trace τ_0, then $\mathcal{M}_0' \cap \mathcal{N}_0 \subset_{\delta_0,\tau_0} \mathcal{Z}(\mathcal{M}_0)$.

Lemma 3.1. *Let $\delta_1 \in (0,1)$ and consider an inclusion $\mathcal{M}_1 \subseteq \mathcal{N}_1$, where $\mathcal{N}_1$ has a unital faithful normal trace τ_1 relative to which $\mathcal{N}_1 \subset_{\delta_1,\tau_1} \mathcal{M}_1$. Then there exists a projection $p_1 \in \mathcal{Z}(\mathcal{M}_1' \cap \mathcal{N}_1)$ such that $\tau_1(p_1) \geq 1 - \delta_1^2$ and $(\mathcal{M}_1 p_1)' \cap p_1\mathcal{N}_1 p_1$ is abelian.*

Setting $\mathcal{M}_2 = \mathcal{M}_1 p_1$, $\mathcal{N}_2 = p_1\mathcal{N}_1 p_1$, and $\tau_2 = \tau_1(p_1)^{-1}\tau_1$, we have $\mathcal{M}_2' \cap \mathcal{N}_2$ is abelian and $\mathcal{N}_2 \subset_{\delta_2,\tau_2} \mathcal{M}_2$, where δ_2 is defined by $\delta_2^2 = \delta_1^2(1-\delta_1^2)^{-1}$.

Proof. Let $\mathcal{C} = \mathcal{M}_1' \cap \mathcal{N}_1$, which contains $\mathcal{Z}(\mathcal{M}_1)$. If $c \in \mathcal{C}$, $\|c\| \leq 1$, we may choose $m \in \mathcal{M}_1$ to satisfy $\|c - m\|_{2,\tau_1} \leq \delta_1$. Conjugation by unitaries from $\mathcal{M}_1$ leaves c invariant, so Dixmier's approximation theorem, [4], shows that there is an element $z \in \mathcal{Z}(\mathcal{M}_1)$ such that $\|c - z\|_{2,\tau_1} \leq \delta_1$. Thus $\mathcal{C} \subset_{\delta_1} \mathcal{Z}(\mathcal{M}_1)$. Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{C}$ which contains $\mathcal{Z}(\mathcal{M}_1)$, and note that $\mathcal{Z}(\mathcal{C}) \subseteq \mathcal{A}$. Choose a projection $p_1 \in \mathcal{Z}(\mathcal{C})$, maximal with respect to the property that $\mathcal{C}p_1$ is abelian. We now construct a unitary $u \in \mathcal{C}(1-p_1)$ such that $\mathbb{E}_{\mathcal{A}(1-p_1)}(u) = 0$.

The algebra $\mathcal{C}(1-p_1)$ may be decomposed as a direct sum

$$\mathcal{C}(1-p_1) = \bigoplus_{k\geq 0} \mathcal{C}_k, \tag{3.2}$$

where $\mathcal{C}_0$ is type II_1 and each $\mathcal{C}_k$ for $k \geq 1$ has the form $\mathbb{M}_{n_k} \otimes \mathcal{A}_k$ for an abelian subalgebra $\mathcal{A}_k$ of $\mathcal{A}$. Let q_k, $k \geq 0$, be the identity element of $\mathcal{C}_k$. Then $\mathcal{A}q_k$, $k \geq 1$, contains $\mathcal{A}_k$ and is maximal abelian in $\mathcal{C}_k$, so has the form $\mathcal{D}_k \otimes \mathcal{A}_k$ for some diagonal algebra $\mathcal{D}_k \subseteq \mathbb{M}_{n_k}$, [9]. Note that the choice of p_1 implies $n_k \geq 2$. For $k \geq 1$, let u_k be a unitary in $\mathbb{M}_{n_k}$ ($\cong \mathbb{M}_{n_k} \otimes 1$) which cyclically permutes the basis for $\mathcal{D}_k$. For $k = 0$, choose two equivalent orthogonal projections in $\mathcal{A}q_0$ which sum to q_0, let $v \in \mathcal{C}_0$ be an implementing partial isometry, and let $u_0 = v + v^*$. Then $u = \sum_{k=0}^{\infty} u_k$ is a unitary in $\mathcal{C}(1-p_1)$ for which $\mathbb{E}_{\mathcal{A}(1-p_1)}(u) = 0$. Thus $w = p_1 + u$ is a unitary in $\mathcal{C}$. Then

$$\begin{aligned}\delta_1 &\geq \|w - \mathbb{E}_{\mathcal{Z}(\mathcal{M}_1)}(w)\|_{2,\tau_1} \geq \|w - \mathbb{E}_{\mathcal{A}}(w)\|_{2,\tau_1} \\ &= \|u - \mathbb{E}_{\mathcal{A}(1-p_1)}(u)\|_{2,\tau_1} = \|u\|_{2,\tau_1} = \|1-p_1\|_{2,\tau_1},\end{aligned} \tag{3.3}$$

and the inequality $\tau_1(p_1) \geq 1 - \delta_1^2$ follows.

Now let $\mathcal{M}_2 = \mathcal{M}_1 p_1$, $\mathcal{N}_2 = p_1 \mathcal{N}_1 p_1$ and $\tau_2 = \tau_1(p_1)^{-1}\tau_1$. Then $\mathcal{M}_2' \cap \mathcal{N}_2 = \mathcal{C}p_1$, which is abelian, and $\mathcal{N}_2 \subset_{\delta_2,\tau_2} \mathcal{M}_2$, where $\delta_2 = \delta_1(1-\delta_1^2)^{-1/2}$. □

Lemma 3.2. *Let $\delta_2 \in (0, 2^{-1})$ and consider an inclusion $\mathcal{M}_2 \subseteq \mathcal{N}_2$, where $\mathcal{N}_2$ has a unital faithful normal trace τ_2 relative to which $\mathcal{N}_2 \subset_{\delta_2,\tau_2} \mathcal{M}_2$. Further suppose that $\mathcal{M}_2' \cap \mathcal{N}_2$ is abelian. Then there exists a projection $p_2 \in \mathcal{M}_2' \cap \mathcal{N}_2$ such that $\tau_2(p_2) \geq 1 - 4\delta_2^2$ and $(\mathcal{M}_2 p_2)' \cap (p_2 \mathcal{N}_2 p_2) = \mathcal{Z}(\mathcal{M}_2 p_2)$. In particular, when $\mathcal{N}_2$ is abelian we have $\mathcal{N}_2 p_2 = \mathcal{M}_2 p_2$.*

Setting $\mathcal{M}_3 = \mathcal{M}_2 p_2$, $\mathcal{N}_3 = p_2 \mathcal{M}_2 p_2$ and $\tau_3 = \tau_2(p_2)^{-1}\tau_2$, we have $\mathcal{M}_3' \cap \mathcal{N}_3 = \mathcal{Z}(\mathcal{M}_3)$ and $\mathcal{N}_3 \subset_{\delta_3,\tau_3} \mathcal{M}_3$, where δ_3 is defined by $\delta_3^2 = \delta_2^2(1-4\delta_2^2)^{-1}$.

Proof. Let $\mathcal{A} = \mathcal{Z}(\mathcal{M}_2)$ and let $\mathcal{C} = \mathcal{M}_2' \cap \mathcal{N}_2$, which is abelian by hypothesis. It is easy to see that $\mathcal{C} \subset_{\delta_2,\tau_2} \mathcal{A}$, by applying Dixmier's approximation theorem, [4]. Consider the basic construction $\mathcal{A} \subseteq \mathcal{C} \subseteq \langle \mathcal{C}, \mathcal{A} \rangle$ with canonical trace Tr on $\langle \mathcal{C}, \mathcal{A} \rangle$ given by $\mathrm{Tr}(x e_{\mathcal{A}} y) = \tau_2(xy)$ for $x, y \in \mathcal{C}$. Note that $\mathcal{C}$ is maximal abelian in $B(L^2(\mathcal{C}, \tau_2))$ and thus maximal abelian in $\langle \mathcal{C}, \mathcal{A} \rangle$. Following the notation of Proposition 2.4, let h be the element of minimal $\|\cdot\|_{2,\mathrm{Tr}}$-norm in $\overline{K}_{\mathcal{C}}^{w}(e_{\mathcal{A}})$ and recall from (2.35) that $\|h - e_{\mathcal{A}}\|_{2,\mathrm{Tr}} \leq \delta_2$. For each $\lambda \in (2^{-1}, 1)$, let f_λ be the spectral projection of h for the interval $[\lambda, 1]$. Since $h \in \mathcal{C}' \cap \langle \mathcal{C}, \mathcal{A} \rangle = \mathcal{C}$, we also have that $f_\lambda \in \mathcal{C}$ for $2^{-1} < \lambda < 1$. Fix an arbitrary λ in this interval.

We first show that for every projection $q \leq f_\lambda$, the inequality

$$\mathbb{E}_{\mathcal{A}}(q) \geq \lambda \, \mathrm{supp}(\mathbb{E}_{\mathcal{A}}(q)) \tag{3.4}$$

holds. If not, then there exists a projection $q \leq f_\lambda$ and $\varepsilon > 0$ so that the spectral projection q_0 of $\mathbb{E}_{\mathcal{A}}(q)$ for the interval $[0, \lambda - \varepsilon]$ is non–zero. Then

$$0 \neq \mathbb{E}_{\mathcal{A}}(qq_0) \leq \lambda - \varepsilon. \tag{3.5}$$

From this it follows that $e_{\mathcal{A}}(qq_0)e_{\mathcal{A}} \leq (\lambda - \varepsilon)e_{\mathcal{A}}$, which implies that $qq_0e_{\mathcal{A}}qq_0 \leq (\lambda - \varepsilon)qq_0$. (To see this, note that, for any pair of projections e and f, the inequalities $efe \leq \lambda e$, $\|ef\| \leq \sqrt{\lambda}$, $\|fe\| \leq \sqrt{\lambda}$, and $fef \leq \lambda f$ are all equivalent). Averaging this inequality over unitaries in $\mathcal{C}qq_0$, which have the form uqq_0 for unitaries $u \in \mathcal{C}$, leads to

$$hqq_0 \leq (\lambda - \varepsilon)qq_0. \tag{3.6}$$

The inequality $hf_\lambda \geq \lambda f_\lambda$ implies that

$$hqq_0 \geq \lambda qq_0, \tag{3.7}$$

and this contradicts (3.6), establishing (3.4).

Now consider two orthogonal projections q_1 and q_2 in $\mathcal{C}f_\lambda$. From (3.4) we obtain

$$\begin{aligned} 1 &\geq \mathbb{E}_{\mathcal{A}}(q_1 + q_2) \geq \lambda(\text{supp } \mathbb{E}_{\mathcal{A}}(q_1) + \text{supp } \mathbb{E}_{\mathcal{A}}(q_2)) \\ &\geq 2\lambda(\text{supp } \mathbb{E}_{\mathcal{A}}(q_1) \cdot \text{supp } \mathbb{E}_{\mathcal{A}}(q_2)). \end{aligned} \tag{3.8}$$

Since $\lambda > 2^{-1}$, this forces $\mathbb{E}_{\mathcal{A}}(q_1)$ and $\mathbb{E}_{\mathcal{A}}(q_2)$ to have disjoint support projections. Whenever a conditional expectation of one abelian algebra onto another has the property that $\mathbb{E}(p)\mathbb{E}(q) = 0$ for all pairs of orthogonal projections p and q, then $\mathbb{E}$ is the identity. This can be easily seen by considering pairs p and $1 - p$. In our situation, we conclude that $\mathcal{A}f_\lambda = \mathcal{C}f_\lambda$. Let p_2 be the spectral projection of h for the interval $(2^{-1}, 1]$. By taking the limit $\lambda \to 2^{-1}+$, we obtain $\mathcal{A}p_2 = \mathcal{C}p_2$, and the estimate $\tau_2(p_2) \geq 1 - 4\delta_2^2$ follows by taking limits in the inequality

$$\begin{aligned} (1-\lambda)^2(1 - \tau_2(f_\lambda)) &= \tau_2(((1-\lambda)(1-f_\lambda))^2) \leq \tau_2((1-h)^2) \\ &= \text{Tr}(e_{\mathcal{A}}(1-h)^2) = \|e_{\mathcal{A}}(1-h)\|_{2,\text{Tr}}^2 \\ &= \|e_{\mathcal{A}}(e_{\mathcal{A}} - h)\|_{2,\text{Tr}}^2 \leq \|e_{\mathcal{A}} - h\|_{2,\text{Tr}}^2 \leq \delta_2^2. \end{aligned} \tag{3.9}$$

Now let $\mathcal{M}_3 = \mathcal{M}_2p_2$, $\mathcal{N}_3 = p_2\mathcal{N}_2p_2$ and $\tau_3 = \tau_2(p_2)^{-1}\tau_2$. Then $\mathcal{Z}(\mathcal{M}_3) = \mathcal{M}_3' \cap \mathcal{N}_3$ and $\mathcal{N}_3 \subset_{\delta_3,\tau_3} \mathcal{M}_3$, where $\delta_3 = \delta_2(1 - 4\delta_2^2)^{-1/2}$. □

Lemma 3.3. *Let $\delta_3 \in (0, 4^{-1})$ and consider an inclusion $\mathcal{M}_3 \subseteq \mathcal{N}_3$, where $\mathcal{N}_3$ has a unital faithful normal trace τ_3 relative to which $\mathcal{N}_3 \subset_{\delta_3,\tau_3} \mathcal{M}_3$. Further suppose that*

$\mathcal{Z}(\mathcal{M}_3) = \mathcal{M}_3' \cap \mathcal{N}_3$. *Then there exists a projection* $p_3 \in \mathcal{Z}(\mathcal{M}_3)$ *such that* $\tau_3(p_3) \geq 1 - 16\delta_3^2$ *and*

$$\mathcal{Z}(\mathcal{M}_3 p_3) = (\mathcal{M}_3 p_3)' \cap (p_3 \mathcal{N}_3 p_3) = \mathcal{Z}(p_3 \mathcal{N}_3 p_3). \tag{3.10}$$

Setting $\mathcal{M}_4 = \mathcal{M}_3 p_3$, $\mathcal{N}_4 = p_3 \mathcal{N}_3 p_3$ *and* $\tau_4 = \tau_3(p_3)^{-1}\tau_3$, *we have*

$$\mathcal{Z}(\mathcal{M}_4) = \mathcal{M}_4' \cap \mathcal{N}_4 = \mathcal{Z}(\mathcal{N}_4) \tag{3.11}$$

and $\mathcal{N}_4 \subset_{\delta_4,\tau_4} \mathcal{M}_4$, *where* δ_4 *is defined by* $\delta_4^2 = \delta_3^2(1 - 16\delta_3^2)^{-1}$.

Proof. Since $\mathcal{Z}(\mathcal{N}_3) \subseteq \mathcal{M}_3' \cap \mathcal{N}_3$ we have, by hypothesis, that $\mathcal{Z}(\mathcal{N}_3) \subseteq \mathcal{Z}(\mathcal{M}_3)$. If $x \in \mathcal{Z}(\mathcal{M}_3)$, $\|x\| \leq 1$, and u is a unitary in $\mathcal{N}_3$ then choose $m \in \mathcal{M}_3$ such that $\|u - m\|_{2,\tau_3} \leq \delta_3$. It follows that

$$\|ux - xu\|_{2,\tau_3} = \|(u-m)x - x(u-m)\|_{2,\tau_3} \leq 2\delta_3, \tag{3.12}$$

and so $\|uxu^* - x\|_{2,\tau_3} \leq 2\delta_3$. Suitable convex combinations of terms of the form uxu^* converge in norm to an element of $\mathcal{Z}(\mathcal{N}_3)$, showing that $\mathcal{Z}(\mathcal{M}_3) \subset_{2\delta_3,\tau_3} \mathcal{Z}(\mathcal{N}_3)$. Now apply Lemma 3.2 to the inclusion $\mathcal{Z}(\mathcal{N}_3) \subseteq \mathcal{Z}(\mathcal{M}_3)$, taking $\delta_2 = 2\delta_3$. We conclude that there is a projection $p_3 \in \mathcal{Z}(\mathcal{M}_3)$ such that $\tau_3(p_3) \geq 1 - 16\delta_3^2$ and $\mathcal{Z}(\mathcal{N}_3)p_3 = \mathcal{Z}(\mathcal{M}_3)p_3$.

Now let $\mathcal{M}_4 = \mathcal{M}_3 p_3$, $\mathcal{N}_4 = \mathcal{N}_3 p_3$ and $\tau_4 = \tau_3(p_3)^{-1}\tau_3$. Then (3.11) is satisfied and $\mathcal{N}_4 \subset_{\delta_4,\tau_4} \mathcal{M}_4$, where $\delta_4 = \delta_3(1 - 16\delta_3^2)^{-1/2}$. □

Lemma 3.4. *Let* $\delta_4 \in (0, 2^{-1/2})$ *and consider an inclusion* $\mathcal{M}_4 \subseteq \mathcal{N}_4$, *where* $\mathcal{N}_4$ *has a unital faithful normal trace* τ_4 *relative to which* $\mathcal{N}_4 \subset_{\delta_4,\tau_4} \mathcal{M}_4$. *Further suppose that*

$$\mathcal{Z}(\mathcal{M}_4) = \mathcal{M}_4' \cap \mathcal{N}_4 = \mathcal{Z}(\mathcal{N}_4). \tag{3.13}$$

Then there exists a projection $p_4 \in \mathcal{Z}(\mathcal{M}_4)$ *such that* $\tau_4(p_4) \geq 1 - 2\delta_4^2$ *and* $\mathcal{M}_4 p_4 = \mathcal{N}_4 p_4$.

Proof. Consider the basic construction $\mathcal{M}_4 \subseteq \mathcal{N}_4 \subseteq \langle \mathcal{N}_4, \mathcal{M}_4 \rangle$ with associated projection $e_{\mathcal{M}_4}$, and let $h \in \mathcal{N}_4' \cap \langle \mathcal{N}_4, \mathcal{M}_4 \rangle$ be the operator obtained from $e_{\mathcal{M}_4}$ by averaging over the unitary group of $\mathcal{N}_4$. By hypothesis, the conditions of Theorem 2.6 are met, and so $h \in \mathcal{Z}(\mathcal{N}_4)$ and has spectrum contained in $\{1\} \cup [0, 2^{-1}]$. Let $q \in \mathcal{Z}(\mathcal{N}_4) = \mathcal{Z}(\mathcal{M}_4)$ be the spectral projection of h for the eigenvalue 1, and note that $h(1-q) \leq (1-q)/2$. Fix an arbitrary $\varepsilon > 0$ and suppose that

$$\mathrm{Tr}(e_{\mathcal{M}_4} u (e_{\mathcal{M}_4}(1-q)) u^*) \geq (2^{-1} + \varepsilon)\ \mathrm{Tr}(e_{\mathcal{M}_4}(1-q)) \tag{3.14}$$

for all unitaries $u \in \mathcal{M}_4$. Taking the average leads to

$$\mathrm{Tr}(e_{\mathcal{M}_4} h(1-q)) \geq (2^{-1}+\varepsilon)\ \mathrm{Tr}(e_{\mathcal{M}_4}(1-q)), \tag{3.15}$$

and so $\tau_4(h(1-q)) \geq (2^{-1}+\varepsilon)\tau_4(1-q)$. If $q=1$ then we already have $\mathcal{N}_4 = \mathcal{M}_4$; otherwise the last inequality gives a contradiction and so (3.14) fails for every $\varepsilon > 0$. The presence of $(1-q)$ in (3.14) ensures that this inequality fails for a unitary $u_\varepsilon \in \mathcal{N}_4(1-q)$. Thus

$$\mathrm{Tr}(e_{\mathcal{M}_4} u_\varepsilon (e_{\mathcal{M}_4}(1-q)) u_\varepsilon^*) < (2^{-1}+\varepsilon)\ \mathrm{Tr}(e_{\mathcal{M}_4}(1-q)) \tag{3.16}$$

for each $\varepsilon > 0$. Define a unitary in $\mathcal{N}_4$ by $v_\varepsilon = q + u_\varepsilon$. By hypothesis,

$$\begin{aligned}
\delta_4^2 &\geq \|q+u_\varepsilon - \mathbb{E}_{\mathcal{M}_4}(q+u_\varepsilon)\|_{2,\tau_4}^2 = \|(I-\mathbb{E}_{\mathcal{M}_4})(u_\varepsilon)\|_{2,\tau_4}^2 \\
&= \|u_\varepsilon\|_{2,\tau_4}^2 - \|\mathbb{E}_{\mathcal{M}_4}(u_\varepsilon)\|_{2,\tau_4}^2 = \tau_4(1-q) - \tau_4(\mathbb{E}_{\mathcal{M}_4}(u_\varepsilon)u_\varepsilon^*) \\
&= \tau_4(1-q) - \mathrm{Tr}(e_{\mathcal{M}_4} u_\varepsilon e_{\mathcal{M}_4}(1-q)u_\varepsilon^*) \\
&\geq \tau_4(1-q) - (2^{-1}+\varepsilon)\tau_4(1-q),
\end{aligned} \tag{3.17}$$

where we have used (3.16) and the fact that $q \in \mathcal{Z}(\mathcal{M}_4) = \mathcal{Z}(\langle \mathcal{N}_4, \mathcal{M}_4 \rangle)$. Letting $\varepsilon \to 0$ in (3.17), we obtain $\tau_4(q) \geq 1 - 2\delta_4^2$.

Define $p_4 = q \in \mathcal{Z}(\mathcal{M}_4)$. The basic construction for $\mathcal{M}_4 p_4 \subseteq \mathcal{N}_4 p_4$ is obtained from the basic construction for $\mathcal{M}_4 \subseteq \mathcal{N}_4$ by cutting by the central projection q. Since $hq = q$, it follows that $\mathcal{N}_4 p_4 = \mathcal{M}_4 p_4$, completing the proof. □

We now summarize these lemmas.

Theorem 3.5. *Let $\mathcal{N}$ be a von Neumann algebra with a unital faithful normal trace τ, let $\mathcal{M}$ be a von Neumann subalgebra, and let δ be a positive number in the interval $(0, (23)^{-1/2})$. If $\mathcal{N} \subset_{\delta,\tau} \mathcal{M}$, then there exists a projection $p \in \mathcal{Z}(\mathcal{M}' \cap \mathcal{N})$ such that $\tau(p) \geq 1 - 23\delta^2$ and $\mathcal{M}p = p\mathcal{N}p$.*

Proof. We apply the previous four lemmas successively to cut by projections until the desired conclusion is reached. Each projection has trace at least a fixed proportion of the trace of the previous one, so the estimates in these lemmas combine to give

$$\tau(p) \geq (1-\delta_1^2)(1-4\delta_2^2)(1-16\delta_3^2)(1-2\delta_4^2), \tag{3.18}$$

where the δ_i's satisfy the relations

$$\delta_1^2 = \delta^2, \quad \delta_2^2 = \frac{\delta_1^2}{1-\delta_1^2}, \quad \delta_3^2 = \frac{\delta_2^2}{1-4\delta_2^2}, \quad \delta_4^2 = \frac{\delta_3^2}{1-16\delta_3^2}. \tag{3.19}$$

Substitution of (3.19) into (3.18) gives $\tau(p) \geq 1 - 23\delta^2$. □

Remark 3.6. The assumption that $\delta < (23)^{-1/2}$ in Theorem 3.5 guarantees that the δ_i's in the lemmas fall in the correct ranges. This theorem is still true, but vacuous, for $\delta \geq (23)^{-1/2}$. The constant 23 can be improved under additional hypotheses by joining the sequence of lemmas at a later point. If the inclusion $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{N} \subset_\delta \mathcal{M}$ also satisfies the hypotheses of Lemmas 3.2, 3.3 or 3.4 then 23 can be replaced respectively by 22, 18 or 2. □

For the case when the larger algebra is a factor, the estimate in Theorem 3.5 can be considerably improved.

Theorem 3.7. *Let $\mathcal{N}$ be a type II_1 factor with a unital faithful normal trace τ, let $\mathcal{M}$ be a von Neumann subalgebra, and let δ be a positive number in the interval $(0, (2/5)^{1/2})$. If $\mathcal{N} \subset_{\delta,\tau} \mathcal{M}$, then there exists a projection $p \in \mathcal{M}' \cap \mathcal{N}$ with $\tau(p) \geq 1 - \delta^2/2$ such that $\mathcal{M}p = p\mathcal{N}p$.*

Proof. Let $q \in \mathcal{M}' \cap \mathcal{N}$ be a projection with $\tau(q) \geq 1/2$. Then $1 - q$ is equivalent in $\mathcal{N}$ to a projection $e \leq q$. Let $v \in \mathcal{N}$ be a partial isometry such that $vv^* = e$, $v^*v = 1 - q$. Let $w = v + v^* \in \mathcal{N}$ and note that $\|w\| = 1$ and $\mathbb{E}_{\mathcal{M}}(w) = 0$. Then

$$\|w - \mathbb{E}_{\mathcal{M}}(w)\|_{2,\tau}^2 = \|v + v^*\|_2^2 = 2\tau(1 - q), \tag{3.20}$$

so we must have $2\tau(1 - q) \leq \delta^2$, or $\tau(q) \geq 1 - \delta^2/2$. On the other hand, if $\tau(q) \leq 1/2$ then this argument applies to $1 - q$, giving $\tau(q) \leq \delta^2/2$. Thus the range of the trace on projections in $\mathcal{M}' \cap \mathcal{N}$ is contained in $[0, \delta^2/2] \cup [1 - \delta^2/2, 1]$.

By Zorn's lemma and the normality of the trace, there is a projection $p \in \mathcal{M}' \cap \mathcal{N}$ which is minimal with respect to the property of having trace at least $1 - \delta^2/2$. We now show that p is a minimal projection in $\mathcal{M}' \cap \mathcal{N}$. If not, then p can be written $p_1 + p_2$ with $\tau(p_1)$, $\tau(p_2) > 0$. By choice of p, we see that $\tau(p_1)$, $\tau(p_2) \leq \delta^2/2$. It follows that

$$1 - \delta^2/2 \leq \tau(p) = \tau(p_1) + \tau(p_2) \leq \delta^2, \tag{3.21}$$

which contradicts $\delta^2 < 2/5$. Thus p is minimal in $\mathcal{M}' \cap \mathcal{N}$, so $\mathcal{M}p$ has trivial relative commutant in $p\mathcal{N}p$. Let τ_1 be the normalized trace $\tau(p)^{-1}\tau$ on $p\mathcal{N}p$. Then $p\mathcal{N}p \subset_{\delta_1,\tau_1} \mathcal{M}p$, where $\delta_1 = \delta(1 - \delta^2/2)^{-1/2} < 2^{-1/2}$.

We have now reached the situation of a subfactor inclusion $\mathcal{P} \subseteq \mathcal{Q}$, $\mathcal{Q} \subset_\delta \mathcal{P}$ for a fixed $\delta < 2^{-1/2}$ and $\mathcal{P}' \cap \mathcal{Q} = \mathbb{C}1$. Since

$$\mathcal{Q}' \cap \langle \mathcal{Q}, \mathcal{P} \rangle = J(\mathcal{P}' \cap \mathcal{Q})J = \mathbb{C}1, \tag{3.22}$$

the operator h obtained from averaging $e_{\mathcal{P}}$ over unitaries in $\mathcal{Q}$ is $\lambda 1$ for some $\lambda > 0$. By Proposition 2.4 (ii) and (iii), we have $1 - \lambda \leq \delta^2 < 1/2$ and $\lambda = \lambda^2 \mathrm{Tr}(1)$, yielding $\mathrm{Tr}(1) = 1/\lambda < 2$. Thus $[\mathcal{Q} \colon \mathcal{P}] < 2$, so $\mathcal{Q} = \mathcal{P}$ from [7]. Applying this to $\mathcal{M}p \subseteq p\mathcal{N}p$, we conclude equality as desired. $\square$

4 Estimates in the $\|\cdot\|_2$-norm

This section establishes some more technical results which will be needed subsequently. Throughout $\mathcal{N}$ is a finite von Neumann algebra with a unital faithful normal trace τ and $\mathcal{A}$ is a general von Neumann subalgebra.

Lemma 4.1. *Let $w \in \mathcal{N}$ have polar decomposition $w = vk$, where $k = (w^*w)^{1/2}$, and let $p = v^*v$ and $q = vv^*$ be the initial and final projections of v. If $e \in \mathcal{N}$ is a projection satisfying $ew = w$, then*

(i) $\|p - k\|_2 \leq \|e - w\|_2$; (4.1)

(ii) $\|e - q\|_2 \leq \|e - w\|_2$; (4.2)

(iii) $\|e - v\|_2 \leq 2\|e - w\|_2$. (4.3)

Proof. The first inequality is equivalent to

$$\tau(p + k^2 - 2pk) \leq \tau(e + w^*w - w - w^*), \tag{4.4}$$

since $ew = w$, and (4.4) is in turn equivalent to

$$\tau(w + w^*) \leq \tau(e - p + 2k), \tag{4.5}$$

since $pk = k$ from properties of the polar decomposition. The map $x \mapsto \tau(k^{1/2}xk^{1/2})$ is a positive linear functional whose norm is $\tau(k)$. Thus

$$|\tau(w)| = |\tau(w^*)| = |\tau(vk)| = |\tau(k^{1/2}vk^{1/2})| \leq \tau(k). \tag{4.6}$$

The range of e contains the range of w, so $e \geq q$. Thus

$$\tau(e) \geq \tau(q) = \tau(p), \tag{4.7}$$

and so (4.5) follows from (4.6), establishing (i).

The second inequality is equivalent to

$$\begin{aligned}\tau(q - 2eq) &\leq \tau(w^*w - ew - w^*e)\\ &= \tau(k^2 - w - w^*).\end{aligned} \tag{4.8}$$

Since $eq = q$, this is equivalent to

$$\tau(w + w^*) \leq \tau(k^2 + q) = \tau(p + k^2). \tag{4.9}$$

From (4.6)

$$\tau(w+w^*) \le 2\ \tau(k) = \tau(p+k^2-(k-p)^2) \le \tau(p+k^2), \tag{4.10}$$

which establishes (4.9) and proves (ii).

The last inequality is

$$\begin{aligned}\|e-v\|_2 &\le \|e-vk\|_2 + \|v(k-p)\|_2\\ &\le \|e-w\|_2 + \|k-p\|_2 \le 2\|e-w\|_2,\end{aligned} \tag{4.11}$$

by (i). □

The next result gives some detailed properties of the polar decomposition (see [2]).

Lemma 4.2. *Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{N}$ and let $\phi\colon\ \mathcal{A} \to \mathcal{N}$ be a normal $*$-homomorphism. Let w have polar decomposition*

$$w = v(w^*w)^{1/2} = (ww^*)^{1/2}v, \tag{4.12}$$

and let $p = v^*v, \quad q = vv^*$. If

$$\phi(a)w = wa, \qquad a \in \mathcal{A}, \tag{4.13}$$

then

$$\text{(i)} \quad w^*w \in \mathcal{A}' \quad \text{and} \quad ww^* \in \phi(\mathcal{A})'; \tag{4.14}$$

$$\text{(ii)} \quad \phi(a)v = va \quad \text{and} \quad \phi(a)q = vav^* \text{ for all } \quad a \in \mathcal{A}; \tag{4.15}$$

$$\text{(iii)} \quad p \in \mathcal{A}' \cap \mathcal{N} \quad \text{and} \quad q \in \phi(\mathcal{A})' \cap \mathcal{N}. \tag{4.16}$$

Proof. If $a \in \mathcal{A}$, then

$$\begin{aligned}w^*wa &= w^*\phi(a)w = (\phi(a^*)w)^*w\\ &= (wa^*)^*w = aw^*w,\end{aligned} \tag{4.17}$$

and so $w^*w \in \mathcal{A}'$. The second statement in (i) has a similar proof.

Let f be the projection onto the closure of the range of $(w^*w)^{1/2}$. Since $w^* = (w^*w)^{1/2}v^*$, the range of w^* is contained in the range of f, and so $f \ge p$ by Lemma 2.3(i). For all $x \in \mathcal{N}$ and $a \in \mathcal{A}$,

$$\begin{aligned}\phi(a)v(w^*w)^{1/2}x &= \phi(a)wx = wax\\ &= v(w^*w)^{1/2}ax = va(w^*w)^{1/2}x,\end{aligned} \tag{4.18}$$

since $(w^*w)^{1/2} \in \mathcal{A}'$ by (i). Thus

$$\phi(a)vf = vaf, \qquad a \in \mathcal{A}, \tag{4.19}$$

which reduces to

$$\phi(a)v = va, \qquad a \in \mathcal{A}, \tag{4.20}$$

since $f \in VN((w^*w)^{1/2}) \subseteq \mathcal{A}'$, and

$$v = vp = vpf = vf. \tag{4.21}$$

This proves the first statement in (ii). The second is immediate from

$$\phi(a)q = \phi(a)vv^* = vav^*, \qquad a \in \mathcal{A}. \tag{4.22}$$

The proof of the third part is similar to that of the first, and we omit the details. □

5 Homomorphisms on subalgebras

In this section we consider two close subalgebras $\mathcal{B}_0$ and $\mathcal{B}$ of a type II_1 factor $\mathcal{N}$. Our objective is to cut each algebra by a projection of large trace in such a way that the resulting algebras are spatially isomorphic by a partial isometry which is close to the identity. The proof of this involves unbounded operators on $L^2(\mathcal{N},\tau)$, so we begin with a brief discussion of those operators which will appear below. A general reference for the basic facts about unbounded operators is [10, section 5.6].

When $1 \in \mathcal{N}$ is viewed as an element of $L^2(\mathcal{N},\tau)$ we will denote it by ξ, and then $x\xi$ is the vector in $L^2(\mathcal{N},\tau)$ corresponding to $x \in \mathcal{N}$. We then have a dense subspace $\mathcal{N}\xi \subseteq L^2(\mathcal{N},\tau)$. For each $\eta \in L^2(\mathcal{N},\tau)$, we may define a linear operator ℓ_η with domain $\mathcal{N}\xi$ by

$$\ell_\eta(x\xi) = Jx^*J\eta, \quad x \in \mathcal{N}. \tag{5.1}$$

If η happens to be $y\xi$ for some $y \in \mathcal{N}$, then ℓ_η coincides with y, but in general ℓ_η is unbounded. For $x, y \in \mathcal{N}$, we have

$$\begin{aligned} \langle \ell_\eta x\xi, y\xi\rangle &= \langle Jx^*J\eta, y\xi\rangle = \overline{\langle x^*J\eta, Jy\xi\rangle} \\ &= \langle JyJ\xi, x^*J\eta\rangle = \langle x\xi, Jy^*JJ\eta\rangle, \end{aligned} \tag{5.2}$$

and so ℓ_η has a densely defined adjoint which agrees with $\ell_{J\eta}$ on $\mathcal{N}\xi$. Thus each ℓ_η is closable, and we denote the closure by L_η. The operators that we consider will all have domains containing $\mathcal{N}\xi$, so it will be convenient to adopt the notation $S \doteq T$ to mean that S and T agree on $\mathcal{N}\xi$. This frees us from having to identify precisely the domain of each particular operator. We note that all unbounded operators arising in the next result are affiliated with $\mathcal{N}$, and thus any bounded operators obtained from the functional calculus will lie in $\mathcal{N}$.

The following lemma will form part of the proof of Theorem 5.2. Looking ahead, we will need to draw certain conclusions from (5.29); for reasons of technical simplicity we consider the adjoint of this equation below.

Lemma 5.1. *Let $\mathcal{B}_0$ and $\mathcal{B}$ be von Neumann subalgebras of $\mathcal{N}$, and let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{B}_0$ whose identity is not assumed to be that of $\mathcal{B}_0$. Suppose that there exist $W \in \langle \mathcal{N}, \mathcal{B}\rangle$ and a *-homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ such that $aW = W\theta(a)$ for $a \in \mathcal{A}$ and such that $We_{\mathcal{B}} = W$. Then there exists a partial isometry $w \in \mathcal{N}$ with the following properties:*

(i) $w^*aw = \theta(a)w^*w, \quad a \in \mathcal{A}$;

(ii) $\|1 - w\|_{2,\tau} \leq 2\|e_{\mathcal{B}} - W\|_{2,Tr}$;

(iii) $\|1 - p'\|_{2,\tau} \leq \|e_{\mathcal{B}} - W\|_{2,Tr}$, *where* $p' = w^*w \in \theta(\mathcal{A})' \cap \mathcal{N}$;

(iv) *if* $q_1 \in \mathcal{N}$ *and* $q_2 \in \mathcal{B}$ *are projections such that* $q_1W = W = Wq_2$ *then* $q_1w = w = wq_2$.

Proof. Let $\eta = W\xi \in L^2(\mathcal{N}, \tau)$. We will first show that $aL_\eta \doteq L_\eta\theta(a)$ for $a \in \mathcal{A}$. Since span$\{\mathcal{N}e_{\mathcal{B}}\mathcal{N}\}$ is weakly dense in $\langle \mathcal{N}, \mathcal{B}\rangle$, we may choose, by the Kaplansky density theorem, a sequence $\{y_n\}_{n=1}^\infty$ from span$\{\mathcal{N}e_{\mathcal{B}}\mathcal{N}\}$ converging to W in the strong* topology. Since $W = We_{\mathcal{B}}$, we also have that $y_ne_{\mathcal{B}} \to W$ in this topology, and each $y_ne_{\mathcal{B}}$ has the form $w_ne_{\mathcal{B}}$ for $w_n \in \mathcal{N}$, since $e_{\mathcal{B}}\mathcal{N}e_{\mathcal{B}} = \mathcal{B}e_{\mathcal{B}}$ by Lemma 2.1. We also note that $e_{\mathcal{B}}$ commutes with each $\theta(a) \in \mathcal{B}$, and that $e_{\mathcal{B}}\xi = \xi$. Then, for $a \in \mathcal{A}$, $x \in \mathcal{N}$,

$$\begin{aligned} aL_\eta x\xi &= aJx^*J\eta = Jx^*JaWe_{\mathcal{B}}\xi \\ &= Jx^*JWe_{\mathcal{B}}\theta(a)\xi = \lim_{n\to\infty} Jx^*Jw_ne_{\mathcal{B}}\theta(a)\xi \\ &= \lim_{n\to\infty} Jx^*Jw_nJ\theta(a^*)J\xi = \lim_{n\to\infty} Jx^*JJ\theta(a^*)Jw_ne_{\mathcal{B}}\xi \\ &= Jx^*JJ\theta(a^*)JW\xi = L_\eta\theta(a)x\xi, \end{aligned} \tag{5.3}$$

establishing that $aL_\eta \doteq L_\eta\theta(a)$. Let $T = |L_\eta|$, and let $L_\eta = wT$ be the polar decomposition of L_η, where w is a partial isometry mapping the closure of the range of T to the closure of the range of L_η. Then

$$awT \doteq wT\theta(a), \quad a \in \mathcal{A}. \tag{5.4}$$

Let $p' = w^*w$, the projection onto the closure of the range of T. Then $p'T \doteq T$ and (5.4) becomes

$$w^*awT \doteq T\theta(a), \quad a \in \mathcal{A}. \tag{5.5}$$

For each $n \in \mathbb{N}$, let $e_n \in \mathcal{N}$ be the spectral projection of T for the interval $[0, n]$. Then each e_n commutes with T and so we may multiply on both sides of (5.5) by e_n to obtain

$$e_nw^*awe_ne_nTe_n = e_nTe_ne_n\theta(a)e_n, \quad a \in \mathcal{A}, \tag{5.6}$$

where $e_nTe_n \in \mathcal{N}$ is now a bounded operator. When $a \geq 0$, (5.6) implies that $(e_nTe_n)^2$ commutes with $e_n\theta(a)e_n$, and thus so also does e_nTe_n. It follows that

$$e_nw^*awTe_n = e_n\theta(a)Te_n, \quad a \in \mathcal{A}, \; n \geq 1. \tag{5.7}$$

For $m \le n$, we can multiply on the left by e_m and then let $n \to \infty$ to obtain that

$$e_m w^* a w \zeta = e_m \theta(a)\zeta, \quad a \in \mathcal{A},\ \zeta \in \operatorname{Ran} T. \tag{5.8}$$

Now let $m \to \infty$ to deduce that w^*aw and $\theta(a)$ agree on $\operatorname{Ran} T$, and consequently that

$$w^*aw = w^*aww^*w = \theta(a)w^*w = \theta(a)p', \quad a \in \mathcal{A}, \tag{5.9}$$

since $p' = 1$ on $\operatorname{Ran} T$. This establishes (i), and by taking $a \ge 0$ in (5.9), it is clear that $p' \in \theta(\mathcal{A})' \cap \mathcal{N}$. We now turn to the norm estimates of (ii) and (iii).

On $[0,\infty)$, define continuous functions f_n for $n \ge 1$ by $f_n(t) = \chi_{[0,n]}(t) + nt^{-1}\chi_{(n,\infty)}(t)$, and let $h_n = f_n(T) \in \mathcal{N}$ be the associated operators arising from the functional calculus. These functions were chosen to have the following properties: they form an increasing sequence with pointwise limit 1, each f_n dominates a positive multiple of each $\chi_{[0,m]}$, and each $tf_n(t)$ is a bounded function. Thus $\{h_n\}_{n=1}^{\infty}$ increases strongly to 1 and each Th_n is a bounded operator and thus in $\mathcal{N}$. The range of each h_n contains the range of each e_m, and so $L_\eta h_n$ and L_η have identical closures of ranges for every $n \ge 1$. Thus w is also the partial isometry in the polar decomposition $L_\eta h_n = wTh_n$, $n \ge 1$. The point of introducing the h_n's is to reduce to the case of bounded operators where we can now apply Lemma 4.1 to obtain

$$\|1-w\|_{2,\tau} \le 2\|1-L_\eta h_n\|_{2,\tau}, \quad \|1-p'\|_{2,\tau} \le \|1-L_\eta h_n\|_{2,\tau}, \tag{5.10}$$

for all $n \ge 1$. For each $b \in \mathcal{B}$,

$$L_\eta b\xi = Jb^*JW\xi = \lim_{n\to\infty} Jb^*Jw_n e_{\mathcal{B}}\xi = \lim_{n\to\infty} w_n e_{\mathcal{B}} Jb^*J\xi = Wb\xi, \tag{5.11}$$

using that $e_{\mathcal{B}}$ commutes with both b and J. Thus $L_\eta e_{\mathcal{B}} = We_{\mathcal{B}} = W$ and, since $ww^*L_\eta \doteq L_\eta$, we also have $ww^*W = W$. Returning to (5.10), we obtain

$$\begin{aligned}\|1-L_\eta h_n\|_{2,\tau} &= \|e_{\mathcal{B}} - L_\eta h_n e_{\mathcal{B}}\|_{2,\mathrm{Tr}} = \|e_{\mathcal{B}} - wTh_n e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \\ &= \|e_{\mathcal{B}} - wh_nTe_{\mathcal{B}}\|_{2,\mathrm{Tr}} = \|e_{\mathcal{B}} - wh_nw^*L_\eta e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \\ &= \|e_{\mathcal{B}} - wh_nw^*We_{\mathcal{B}}\|_{2,\mathrm{Tr}}. \end{aligned} \tag{5.12}$$

Since Tr is normal we may let $n \to \infty$ in this last equation, giving $\lim_{n\to\infty} \|1 - L_\eta h_n\|_{2,\tau} = \|e_{\mathcal{B}} - W\|_{2,\mathrm{Tr}}$. The inequalities of (ii) and (iii) now follow by letting $n \to \infty$ in (5.10). We now establish (iv).

Let $q_1 \in \mathcal{N}$ be such that $q_1W = W$. For each $x \in \mathcal{N}$,

$$L_\eta(x\xi) = Jx^*JW\xi = Jx^*Jq_1W\xi = q_1Jx^*JW\xi \tag{5.13}$$

and so q_1 is the identity on the closure of the range of L_η which is also the range of w. Thus $q_1 w = w$.

Now suppose that $q_2 \in \mathcal{B}$ is such that $Wq_2 = W$. By replacing q_2 by $1 - q_2$, we may prove the equivalent statement that $wq_2 = 0$ follows from $Wq_2 = 0$. Then

$$0 = Wq_2 = We_{\mathcal{B}}q_2 = Wq_2 e_{\mathcal{B}} = L_\eta q_2 e_{\mathcal{B}} = wTq_2 e_{\mathcal{B}}. \tag{5.14}$$

Multiply by $e_n w^*$ to obtain $e_n T q_2 e_{\mathcal{B}} = 0$. Since $e_n T \in \mathcal{N}$, it follows from Lemma 2.1 that $e_n T q_2 = 0$ for all $n \geq 1$. Then $q_2 T e_n = 0$ for all $n \geq 1$, and letting n increase, we find that q_2 annihilates the closure of the range of T which is also the range of w^*. Thus $q_2 w^* = 0$ and so $wq_2 = 0$, completing the proof. □

The following is the main result of this section. We will also state two variants which give improved estimates under stronger hypotheses.

Theorem 5.2. *Let $\delta > 0$, let $\mathcal{B}_0$ and $\mathcal{B}$ be von Neumann subalgebras of a type* II_1 *factor $\mathcal{N}$ with unital faithful normal trace τ, and suppose that $\|\mathbb{E}_{\mathcal{B}} - \mathbb{E}_{\mathcal{B}_0}\|_{\infty,2} \leq \delta$. Then there exist projections $q_0 \in \mathcal{B}_0$, $q \in \mathcal{B}$, $q_0' \in \mathcal{B}_0' \cap \mathcal{N}$, $q' \in \mathcal{B}' \cap \mathcal{N}$, $p_0' = q_0 q_0'$, $p' = qq'$, and a partial isometry $v \in \mathcal{N}$ such that $vp_0'\mathcal{B}_0 p_0' v^* = p'\mathcal{B}p'$, $vv^* = p'$, $v^*v = p_0'$. Moreover, v can be chosen to satisfy $\|1 - v\|_{2,\tau} \leq 69\delta$, $\|1 - p'\|_{2,\tau} \leq 35\delta$ and $\|1 - p_0'\|_{2,\tau} \leq 35\delta$.*

Under the additional hypothesis that the relative commutants of $\mathcal{B}_0$ and $\mathcal{B}$ are respectively their centers, the projections may be chosen so that $p_0' \in \mathcal{B}_0$ and $p' \in \mathcal{B}$.

Proof. We assume that $\delta < (35)^{-1}$, otherwise we may take $v = 0$. Let $e_{\mathcal{B}}$ be the Jones projection for the basic construction $\mathcal{B} \subseteq \mathcal{N} \subseteq \langle \mathcal{N}, \mathcal{B} \rangle$ and let $h \in \mathcal{B}_0' \cap \langle \mathcal{N}, \mathcal{B} \rangle$ be the operator obtained from averaging $e_{\mathcal{B}}$ over the unitary group of $\mathcal{B}_0$ (see Proposition 2.4). If we denote by e the spectral projection of h for the interval [1/2,1], then $e \in \mathcal{B}_0' \cap \langle \mathcal{N}, \mathcal{B} \rangle$ and $\|e_{\mathcal{B}} - e\|_{2,\mathrm{Tr}} \leq 2\delta$, by Corollary 2.5. Then

$$e\mathcal{B}_0 = e\mathcal{B}_0 e \subseteq e\langle \mathcal{N}, \mathcal{B} \rangle e. \tag{5.15}$$

Consider $x \in e\langle \mathcal{N}, \mathcal{B} \rangle e$ with $\|x\| \leq 1$. Since $e_{\mathcal{B}} \langle \mathcal{N}, \mathcal{B} \rangle e_{\mathcal{B}} = \mathcal{B} e_{\mathcal{B}}$, there exists $b \in \mathcal{B}$, $\|b\| \leq 1$, such that $e_{\mathcal{B}} x e_{\mathcal{B}} = b e_{\mathcal{B}}$. Then

$$\begin{aligned} \|x - \mathbb{E}_{\mathcal{B}_0}(b)e\|_{2,\mathrm{Tr}}^2 &= \|(x - \mathbb{E}_{\mathcal{B}_0}(b))e\|_{2,\mathrm{Tr}}^2 \\ &= \|(x - \mathbb{E}_{\mathcal{B}_0}(b))ee_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2 + \|(x - \mathbb{E}_{\mathcal{B}_0}(b))e(1 - e_{\mathcal{B}})\|_{2,\mathrm{Tr}}^2 \end{aligned} \tag{5.16}$$

and we estimate these terms separately. For the first, we have

$$\begin{aligned}\|(x-\mathbb{E}_{\mathcal{B}_0}(b))ee_{\mathcal{B}}\|_{2,\mathrm{Tr}} &= \|e(x-\mathbb{E}_{\mathcal{B}_0}(b))e_{\mathcal{B}}\|_{2,\mathrm{Tr}}\\ &\le \|e(x-\mathbb{E}_{\mathcal{B}}(b))e_{\mathcal{B}}\|_{2,\mathrm{Tr}} + \|e(\mathbb{E}_{\mathcal{B}}(b)-\mathbb{E}_{\mathcal{B}_0}(b))e_{\mathcal{B}}\|_{2,\mathrm{Tr}}\\ &\le \|e(x-e_{\mathcal{B}}xe_{\mathcal{B}})e_{\mathcal{B}}\|_{2,\mathrm{Tr}}+\delta\\ &= \|e(e-e_{\mathcal{B}})(xe_{\mathcal{B}})\|_{2,\mathrm{Tr}}+\delta\\ &\le \|e-e_{\mathcal{B}}\|_{2,\mathrm{Tr}}+\delta\le 3\delta. \end{aligned}\tag{5.17}$$

For the second term in (5.16), we have

$$\begin{aligned}\|(x-\mathbb{E}_{\mathcal{B}_0}(b))e(1-e_{\mathcal{B}})\|_{2,\mathrm{Tr}}^2 &= \|(x-\mathbb{E}_{\mathcal{B}_0}(b))e(e-e_{\mathcal{B}})\|_{2,\mathrm{Tr}}^2\\ &\le \|x-\mathbb{E}_{\mathcal{B}_0}(b)\|^2\|e-e_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2\\ &\le 16\delta^2. \end{aligned}\tag{5.18}$$

Substituting (5.17) and (5.18) into (5.16) gives

$$\|x-\mathbb{E}_{\mathcal{B}_0}(b)e\|_{2,\mathrm{Tr}}^2\le 25\delta^2. \tag{5.19}$$

Hence $e\langle\mathcal{N},\mathcal{B}\rangle e\subset_{5\delta,\mathrm{Tr}}\mathcal{B}_0e$. Since $\|e-e_{\mathcal{B}}\|_{2,\mathrm{Tr}}^2\le 4\delta^2$, it follows that

$$1+4\delta^2\ge \mathrm{Tr}(e)\ge 1-4\delta^2. \tag{5.20}$$

If we now define a unital trace on $e\langle\mathcal{N},\mathcal{B}\rangle e$ by $\tau_1=\mathrm{Tr}(e)^{-1}\,\mathrm{Tr}$, then $e\langle\mathcal{N},\mathcal{B}\rangle e\subset_{\varepsilon,\tau_1}\mathcal{B}_0e$ where $\varepsilon=5\delta(1-4\delta^2)^{-1/2}$. By Theorem 3.5, there exists a projection $f\in(\mathcal{B}_0e)'\cap e\langle\mathcal{N},\mathcal{B}\rangle e$ with $\tau_1(f)\ge 1-23\varepsilon^2$ such that $\mathcal{B}_0f=f\langle\mathcal{N},\mathcal{B}\rangle f$ (since $ef=f$).

Let $V\in\langle\mathcal{N},\mathcal{B}\rangle$ be the partial isometry in the polar decomposition of $e_{\mathcal{B}}f$, so that $e_{\mathcal{B}}f=(e_{\mathcal{B}}fe_{\mathcal{B}})^{1/2}V$. The inequality $\|V-e_{\mathcal{B}}\|_{2,\mathrm{Tr}}\le\sqrt{2}\,\|e_{\mathcal{B}}-f\|_{2,\mathrm{Tr}}$ is obtained from [3] or Lemma 2.3 (iv), and we estimate this last quantity. We have

$$\begin{aligned}\|e_{\mathcal{B}}-f\|_{2,\mathrm{Tr}}^2 &= \mathrm{Tr}(e_{\mathcal{B}}+f-2e_{\mathcal{B}}f)\\ &= \mathrm{Tr}(e_{\mathcal{B}}+e-2e_{\mathcal{B}}e-(e-f)+2e_{\mathcal{B}}(e-f))\\ &= \|e_{\mathcal{B}}-e\|_{2,\mathrm{Tr}}^2+\mathrm{Tr}(2e_{\mathcal{B}}(e-f)-(e-f))\\ &\le \|e_{\mathcal{B}}-e\|_{2,\mathrm{Tr}}^2+\mathrm{Tr}(e-f)\le 4\delta^2+23\varepsilon^2\mathrm{Tr}(e)\\ &\le 4\delta^2+(23)(25)\delta^2(1+4\delta^2)/(1-4\delta^2) \end{aligned}\tag{5.21}$$

so $\|V-e_{\mathcal{B}}\|_{2,\mathrm{Tr}}\le\sqrt{2}\,\delta_1$ where δ_1^2 is the last quantity above. Then $VV^*\in e_{\mathcal{B}}\langle\mathcal{N},\mathcal{B}\rangle e_{\mathcal{B}}=\mathcal{B}e_{\mathcal{B}}$, and $V^*V\in f\langle\mathcal{N},\mathcal{B}\rangle f=\mathcal{B}_0f$, by the choice of f. Thus there exist projections $p_0\in\mathcal{B}_0$, $p\in\mathcal{B}$ so that

$$V^*V=p_0f,\qquad VV^*=pe_{\mathcal{B}}. \tag{5.22}$$

If $z \in \mathcal{Z}(\mathcal{B}_0)$ is the central projection corresponding to the kernel of the homomorphism $b_0 \mapsto b_0 f$ on $\mathcal{B}_0$, then by replacing p_0 by $p_0(1-z)$, we may assume that $p_0 b_0 p_0 = 0$ whenever $p_0 b_0 p_0 f = 0$. Since $p_0(1-z)f = p_0 f$, (5.22) remains valid and we note that the following relations (and their adjoints) hold:

$$V = Vp_0 = Vf = pV = e_{\mathcal{B}}V. \tag{5.23}$$

Define $\Theta \colon p_0\mathcal{B}_0 p_0 \to \langle \mathcal{N}, \mathcal{B}\rangle$ by

$$\Theta(b_0) = Vb_0V^*, \quad b_0 \in p_0\mathcal{B}_0p_0. \tag{5.24}$$

We will show that Θ is a *–isomorphism onto $p\mathcal{B}pe_{\mathcal{B}}$. Now $pe_{\mathcal{B}}V = V$ from (5.23), so the range of Θ is contained in $pe_{\mathcal{B}}\langle \mathcal{N}, \mathcal{B}\rangle e_{\mathcal{B}}p = p\mathcal{B}pe_{\mathcal{B}}$. Since

$$V^*\Theta(b_0)V = p_0 f b_0 p_0 f, \quad b_0 \in p_0\mathcal{B}_0p_0, \tag{5.25}$$

from (5.22), the choice of p_0 shows that Θ has trivial kernel. The map is clearly self–adjoint, and we check that it is a homomorphism. For $p_0b_0p_0$, $p_0b_1p_0 \in p_0\mathcal{B}_0p_0$,

$$\begin{aligned}\Theta(p_0b_0p_0)\Theta(p_0b_1p_0) &= Vp_0b_0p_0V^*Vp_0b_1p_0V^* = Vp_0b_0p_0fp_0b_1p_0V^*\\ &= Vfp_0b_0p_0b_1p_0V^* = \Theta(p_0b_0p_0b_1p_0),\end{aligned} \tag{5.26}$$

using $Vf = V$. Finally we show that Θ maps onto $p\mathcal{B}pe_{\mathcal{B}}$. Given $b \in \mathcal{B}$, let

$$x = V^*pbpe_{\mathcal{B}}V = p_0V^*pbpe_{\mathcal{B}}Vp_0 \in p_0f\langle \mathcal{N}, \mathcal{B}\rangle fp_0 = p_0\mathcal{B}_0p_0f. \tag{5.27}$$

Then x has the form $p_0b_0p_0f$ for some $b_0 \in \mathcal{B}_0$. Thus

$$\begin{aligned}\Theta(p_0b_0p_0) &= Vp_0b_0p_0V^* = Vp_0b_0p_0fV^*\\ &= VxV^* = VV^*pbpe_{\mathcal{B}}VV^*\\ &= pe_{\mathcal{B}}pbpe_{\mathcal{B}}pe_{\mathcal{B}} = pbpe_{\mathcal{B}},\end{aligned} \tag{5.28}$$

and this shows surjectivity. Thus $\Theta \colon p_0\mathcal{B}_0p_0 \to p\mathcal{B}pe_{\mathcal{B}}$ is a surjective *–isomorphism, and so can be expressed as $\Theta(p_0b_0p_0) = \theta(p_0b_0p_0)e_{\mathcal{B}}$ where $\theta \colon p_0\mathcal{B}_0p_0 \to p\mathcal{B}p$ is a surjective *–isomorphism. From the definitions of these maps,

$$Vb_0 = \theta(b_0)V, \qquad b_0 \in p_0\mathcal{B}_0p_0. \tag{5.29}$$

If we take the adjoint of this equation then we are in the situation of Lemma 5.1 with $W = V^*$ and $\mathcal{A} = p_0\mathcal{B}_0p_0$. We conclude that there is a partial isometry $v \in \mathcal{N}$ (the w^* of the previous lemma) such that

$$vb_0v^* = \theta(b_0)vv^*, \qquad b_0 \in p_0\mathcal{B}_0p_0. \tag{5.30}$$

Then clearly the projection $p' = vv^*$ commutes with $\theta(p_0\mathcal{B}_0p_0) = p\mathcal{B}p$, and it lies under p from Lemma 5.1 (iv) since $pV = V$. Thus $p' \in (p\mathcal{B}p)' \cap p\mathcal{N}p$.

Now consider the projection $p_0' = v^*v$. Since $Vp_0 = V$, it follows from Lemma 5.1 (iv) that $vp_0 = v$, and thus p_0' lies under p_0. From (5.29), we have

$$p_0'b_0p_0' = v^*\theta(b_0)v, \quad b_0 \in p_0\mathcal{B}_0p_0, \tag{5.31}$$

and the map $b_0 \mapsto v^*\theta(b_0)v$ is a *-homomorphism on $p_0\mathcal{B}_0p_0$. Thus, for all $b_0 \in p_0\mathcal{B}_0p_0$,

$$p_0'b_0(1-p_0')b_0^*p_0' = p_0'b_0b_0^*p_0' - (p_0'b_0p_0')(p_0'b_0^*p_0') = 0, \tag{5.32}$$

from which we deduce that $p_0'b_0(1-p_0') = 0$ and that $p_0' \in (p_0\mathcal{B}_0p_0)'$. This shows that $p_0' \in (p_0\mathcal{B}_0p_0)' \cap p_0\mathcal{N}p_0$.

It remains to estimate $\|1-v\|_{2,\tau}$. Since

$$\|V - e_\mathcal{B}\|_{2,\mathrm{Tr}} \leq \sqrt{2}\ \|e_\mathcal{B} - f\|_{2,\mathrm{Tr}} \leq \sqrt{2}\ \delta_1, \tag{5.33}$$

from (5.21), we obtain

$$\|1-v\|_{2,\tau} = \|1-w\|_{2,\tau} \leq 2\|e_\mathcal{B} - W\|_{2,\mathrm{Tr}} = 2\|e_\mathcal{B} - V\|_{2,\mathrm{Tr}} \leq 2\sqrt{2}\delta_1, \tag{5.34}$$

using Lemma 5.1 (ii). The estimate of Lemma 5.1 (iii) gives

$$\|1-p'\|_{2,\tau} \leq \|e_\mathcal{B} - W\|_{2,\mathrm{Tr}} \leq \sqrt{2}\delta_1, \tag{5.35}$$

while a similar estimate holds for $\|1-p_0'\|_{2,\tau}$ because p' and p_0' are equivalent projections in $\mathcal{N}$.

From the definition of δ_1 and the requirement that $\delta < (35)^{-1}$, we see that

$$8\delta_1^2/\delta^2 \leq (69)^2, \tag{5.36}$$

by evaluating the term $(1+4\delta^2)(1-4\delta^2)^{-1}$ at $\delta = 1/35$. The estimate $\|1-v\|_{2,\tau} \leq 69\delta$ follows. Then

$$\|1-p'\|_{2,\tau} \leq \sqrt{2}\ \delta_1 \leq (69/2)\delta \leq 35\delta \tag{5.37}$$

with a similar estimate for $\|1-p_0'\|_{2,\tau}$. The fact that each projection is a product of a projection from the algebra and one from the relative commutant is clear from the proof. The last statement of the theorem is an immediate consequence of the first part, because now the relative commutants are contained in the algebras. □

The estimates in Theorem 5.2, while general, can be substantially improved in special cases. The next result addresses the case of two close masas.

Theorem 5.3. *Let $\delta > 0$, let $\mathcal{B}_0$ and $\mathcal{B}$ be masas in a type II_1 factor $\mathcal{N}$ with unital faithful normal trace τ, and suppose that $\|\mathbb{E}_{\mathcal{B}}-\mathbb{E}_{\mathcal{B}_0}\|_{\infty,2} \leq \delta$. Then there exists a partial isometry $v \in \mathcal{N}$ such that $v^*v = p_0 \in \mathcal{B}_0$, $vv^* = p \in \mathcal{B}$, and $v\mathcal{B}_0v^* = \mathcal{B}p$. Moreover v can be chosen to satisfy $\|1-v\|_{2,\tau} \leq 30\delta$, $\|1-p\|_{2,\tau} \leq 15\delta$ and $\|1-p_0\|_{2,\tau} \leq 15\delta$.*

Proof. We assume that $\delta < (15)^{-1}$, otherwise we may take $v = 0$. By averaging $e_{\mathcal{B}}$ over $\mathcal{B}_0$, we see that there is a projection $e_0 \in \mathcal{B}_0' \cap \langle \mathcal{N}, \mathcal{B}\rangle$ satisfying $\|e_0 - e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \leq 2\delta$. By Lemma 2.2, there exists a central projection $z \in \langle \mathcal{N}, \mathcal{B}\rangle$ such that ze_0 and $ze_{\mathcal{B}}$ are equivalent projections in $\langle \mathcal{N}, \mathcal{B}\rangle$, and $\|ze_0 - e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \leq 2\delta$. Let $e = ze_0 \in \mathcal{B}_0' \cap \langle \mathcal{N}, \mathcal{B}\rangle$, and consider the inclusion $\mathcal{B}_0 e \subseteq e\langle \mathcal{N}, \mathcal{B}\rangle e$. Let $w \in \langle \mathcal{N}, \mathcal{B}\rangle$ be a partial isometry such that $e = ww^*$, $ze_{\mathcal{B}} = w^*w$. Then $wze_{\mathcal{B}} = w$, and so

$$\mathcal{B}_0 e \subseteq e\langle \mathcal{N}, \mathcal{B}\rangle e = wze_{\mathcal{B}}w^*\langle \mathcal{N}, \mathcal{B}\rangle wze_{\mathcal{B}}w^* \subseteq wze_{\mathcal{B}}\mathcal{B}w^* \tag{5.38}$$

and the latter algebra is abelian. The proof now proceeds exactly as in Theorem 5.2, starting from (5.15) which corresponds to (5.38). The only difference is that having an abelian inclusion allows us to replace the constant 23 in (5.21) and subsequent estimates by 4, using Lemma 3.2. This leads to the required estimates on $\|1-v\|_{2,\tau}$, $\|1-p\|_{2,\tau}$ and $\|1-p_0\|_{2,\tau}$. □

We now consider the case of two close subfactors of $\mathcal{N}$.

Theorem 5.4. *Let $\delta > 0$, let $\mathcal{B}_0$ and $\mathcal{B}$ be subfactors of $\mathcal{N}$ and suppose that $\|\mathbb{E}_{\mathcal{B}} - \mathbb{E}_{\mathcal{B}_0}\|_{\infty,2} \leq \delta$. Then there exist projections $q_0 \in \mathcal{B}_0$, $q \in \mathcal{B}$, $q_0' \in \mathcal{B}_0' \cap \mathcal{N}$, $q' \in \mathcal{B}' \cap \mathcal{N}$, $p_0 = q_0q_0'$, $p = qq'$, and a partial isometry $v \in \mathcal{N}$ such that $vp_0\mathcal{B}_0p_0v^* = p\mathcal{B}p$, $vv^* = p$, $v^*v = p_0$, and*

$$\|1-v\|_{2,\tau} \leq 13\delta, \qquad \tau(p) = \tau(p_0) \geq 1 - 67\delta^2. \tag{5.39}$$

If, in addition, the relative commutants of $\mathcal{B}_0$ and $\mathcal{B}$ are both trivial and $\delta < 67^{-1/2}$, then $\mathcal{B}$ and $\mathcal{B}_0$ are unitarily conjugate in $\mathcal{N}$.

Proof. We assume that $\delta < 67^{-1/2}$, otherwise take $v = 0$. The proof is identical to that of Theorem 5.2 except that we now have an inclusion $e\mathcal{B}_0e \subseteq e\langle \mathcal{N}, \mathcal{B}\rangle e$ of factors. Our choice of δ allows us a strict upper bound of $(2/5)^{-1/2}$ on the ε which appears immediately after (5.20). Thus the estimate of Theorem 3.7 applies, which allows us to replace 23 by 1/2 in (5.21). This gives

$$8\delta_1^2/\delta^2 \leq 145 < 169 \tag{5.40}$$

and the estimates of (5.39) follow.

If the relative commutants are trivial then $p \in \mathcal{B}$ and $p_0 \in \mathcal{B}_0$, so v implements an isomorphism between $p\mathcal{B}p$ and $p_0\mathcal{B}_0p_0$ which then easily extends to unitary conjugacy between $\mathcal{B}$ and $\mathcal{B}_0$. □

Let $\mathcal{R}$ be the hyperfinite type II_1 factor, choose a projection $p \in \mathcal{R}$ with $\tau(p) = 1-\delta$, where δ is small, and let θ be an isomorphism of $p\mathcal{R}p$ onto $(1-p)\mathcal{R}(1-p)$. Let $\mathcal{B}_0 = \{x+\theta(x) \colon\ x \in p\mathcal{R}p\}$ and let $\mathcal{B}$ have a similar definition but using an isomorphism ϕ such that $\theta^{-1}\phi$ is a properly outer automorphism of $p\mathcal{R}p$. Such an example shows that the projections from the relative commutants in Theorem 5.4 cannot be avoided.

These results above suggest that it might be possible to obtain similar theorems for one sided inclusions. By this we mean that if $\mathcal{B}_0 \subset_\delta \mathcal{B}$ then there is a partial isometry which moves some compression of $\mathcal{B}_0$ (preferably large) into $\mathcal{B}$. However the following shows that this cannot be so, even if the two algebras are subfactors with trivial relative commutant in some factor $\mathcal{M}$, and even if we renounce the requirement that the size of the compression be large and merely require the compression to be non–zero. In this respect, note that if there exists a non–zero partial isometry $v \in \mathcal{M}$ such that $v^*v \in \mathcal{B}_0$, $vv^* \in \mathcal{B}$ and $vB_0v^* \subseteq vv^*Bvv^*$, then there would be a unitary $u \in \mathcal{M}$ such that $u\mathcal{B}_0u^* \subseteq \mathcal{B}$. It is this that we will contradict, by exhibiting II_1 subfactors $\mathcal{B}_0, \mathcal{B} \in \mathcal{M}$ with trivial relative commutant and $\mathcal{B}_0 \subset_\delta \mathcal{B}$ for δ arbitrarily small, but with no unitary conjugate of $\mathcal{B}_0$ sitting inside $\mathcal{B}$. The construction, based on [16], is given below.

By [16], for each $\lambda < 1/4$ there exists an inclusion of factors $(\mathcal{N}(\lambda) \subseteq \mathcal{M}(\lambda)) = (\mathcal{N} \subseteq \mathcal{M})$ with Jones index $\lambda^{-1} > 4$, trivial relative commutant and graph $\Gamma_{\mathcal{N},\mathcal{M}} = A_\infty$. (Note that in fact by [20] one can take the ambient factor $\mathcal{M}$ to be $\mathcal{L}(\mathbb{F}_\infty)$, for all $\lambda < 1/4$.) Let $e_0 \in \mathcal{M}$ be a projection such that $\mathbb{E}_{\mathcal{N}}(e_0) = \lambda 1$ and let $\mathcal{N}_1 \subseteq \mathcal{M}$ be a subfactor such that $\mathcal{N}_1 \subseteq \mathcal{N} \subseteq \mathcal{M}$ is the basic construction for $\mathcal{N}_1 \subseteq \mathcal{N}$ with Jones projection e_0. Then choose a subfactor $\mathcal{Q} \subseteq \mathcal{M}$ such that $(1-e_0) \in \mathcal{Q}$ and $(1-e_0)\mathcal{Q}(1-e_0) = \mathcal{N}_1(1-e_0)$. An easy computation shows that $\mathcal{Q} \subset_{\delta(\lambda)} \mathcal{N}_1$, where $\delta(\lambda) = 6\lambda - 4\lambda^2$. Thus, since $\mathcal{N}_1 \subseteq \mathcal{N}$, we get $\mathcal{Q} \subset_{\delta(\lambda)} \mathcal{N}$ as well.

Proposition 5.5. *With the above notation, we have $\mathcal{Q} \subset_{\delta(\lambda)} \mathcal{N}$, with $\delta(\lambda) = 6\lambda - 4\lambda^2$, but there does not exist a unitary $v \in \mathcal{M}$ such that $v\mathcal{Q}v^* \subseteq \mathcal{N}$.*

Proof. Suppose there is a unitary $v \in \mathcal{M}$ such that $v\mathcal{Q}v^* \subseteq \mathcal{N}_1$, and let $\mathcal{N}_0$ be $v^*\mathcal{N}v$. Then $\mathcal{N}_0$ is an intermediate factor for $\mathcal{Q}$. But the irreducible subfactors in the Jones tower of a subfactor with Temperley–Lieb–Jones standard lattice do not have intermediate subfactors (see, for example, [1]), giving a contradiction.

An alternative argument goes as follows. The basic construction extension algebra $\mathcal{Q}' \cap \langle \mathcal{M}, \mathcal{Q} \rangle$ contains the projections $e_{\mathcal{Q}}$ and $e_{\mathcal{N}_0}$, which satisfy $e_{\mathcal{Q}} \leq e_{\mathcal{N}_0}$. Their traces are respectively $\lambda^2/(1-\lambda)^2$ and λ. But the relative commutant $\mathcal{Q}' \cap \langle \mathcal{M}, \mathcal{Q} \rangle$ is isomorphic to $\mathbb{C}^3$ and, of the three minimal projections, the only two traces that are less than 1/2 are

$$\frac{\lambda^2}{(1-\lambda)^2} \text{ and } \frac{\lambda}{1-\lambda}. \tag{5.41}$$

Since $\tau(e_{\mathcal{N}_0}) = \lambda$, the only possibility is to have

$$\tau(e_{\mathcal{N}_0}) = \frac{\lambda^2}{(1-\lambda)^2} + \frac{\lambda}{1-\lambda} = \frac{\lambda}{(1-\lambda)^2}. \tag{5.42}$$

This is, of course, impossible. □

6 Unitary congugates of masas

In this section we apply our previous work on perturbations of subalgebras to the particular situation of a masa and a nearby unitary conjugate of it. The main result of this section is Theorem 6.4. This contains two inequalities which we present separately. Since we will be working with only one unital trace we simplify notation by replacing $\|\cdot\|_{2,\tau}$ by $\|\cdot\|_2$, and we denote by $d(x,S)$ the distance in $\|\cdot\|_2$-norm from an element $x \in \mathcal{N}$ to a subset $S \subseteq \mathcal{N}$.

Recall from [14] that the normalizing groupoid $\mathcal{G}(\mathcal{A})$ of a masa $\mathcal{A}$ in $\mathcal{N}$ is the set of partial isometries $v \in \mathcal{N}$ such that vv^*, $v^*v \in \mathcal{A}$, and $v\mathcal{A}v^* = \mathcal{A}vv^*$. Such a partial isometry v implements a spatial $*$-isomorphism between $\mathcal{A}v^*v$ and $\mathcal{A}vv^*$. By choosing a normal $*$-isomorphism between the abelian algebras $\mathcal{A}(1-v^*v)$ and $\mathcal{A}(1-vv^*)$ (both isomorphic to $L^\infty[0,1]$), we obtain a $*$-automorphism of $\mathcal{A}$ satisfying the hypotheses of Lemma 2.1 of [8]. It follows that v has the form pw^*, where p is a projection in $\mathcal{A}$ and $w \in N(\mathcal{A})$ (this result is originally in [6]). The next result will allow us to relate $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}$ to the distance from u to $N(\mathcal{A})$.

Proposition 6.1. *Let $\mathcal{A}$ be a masa in $\mathcal{N}$, let $u \in \mathcal{N}$ be a unitary and let $\varepsilon_1, \varepsilon_2 > 0$. Suppose that there exists a partial isometry $v \in \mathcal{N}$ such that $v^*v \in \mathcal{A}$, $vv^* \in u\mathcal{A}u^*$, $v\mathcal{A}v^* = u\mathcal{A}u^*vv^*$, and*

$$\|v - \mathbb{E}_{u\mathcal{A}u^*}(v)\|_2 \leq \varepsilon_1, \tag{6.1}$$

$$\|v\|_2^2 \geq 1 - \varepsilon_2^2. \tag{6.2}$$

Then there exists $\tilde{u} \in N(\mathcal{A})$ such that

$$\|u - \tilde{u}\|_2 \leq 2(\varepsilon_1 + \varepsilon_2). \tag{6.3}$$

Proof. Let v_1 be the partial isometry $u^*v \in \mathcal{N}$. From the hypotheses we see that $v_1^*v_1$, $v_1v_1^* \in \mathcal{A}$ and

$$v_1\mathcal{A}v_1^* = u^*v\mathcal{A}v^*u = u^*u\mathcal{A}u^*vv^*u = \mathcal{A}v_1v_1^*, \tag{6.4}$$

and so $v_1 \in \mathcal{G}(\mathcal{A})$. It follows from [8] that $v_1 = pw^*$ for some projection $p \in \mathcal{A}$ and unitary $w^* \in N(\mathcal{A})$. Thus

$$vw = up. \tag{6.5}$$

From (6.1), there exists $a \in \mathcal{A}$ such that $\|a\| \leq 1$ and $\mathbb{E}_{u\mathcal{A}u^*}(v) = uau^*$. Since $\mathcal{A}$ is abelian, it is isomorphic to $C(\Omega)$ for some compact Hausdorff space Ω. Writing $b = |a|$, $0 \leq b \leq 1$, there exists a unitary $s \in \mathcal{A}$ such that $a = bs$.

Now (6.2) and (6.5) imply that

$$\|p\|_2^2 = \|v\|_2^2 \geq 1 - \varepsilon_2^2, \tag{6.6}$$

and so

$$\|1 - p\|_2 = (1 - \|p\|_2^2)^{1/2} \leq \varepsilon_2. \tag{6.7}$$

It now follows from (6.5) that

$$\|v - uw^*\|_2 = \|vw - u\|_2 = \|up - u\|_2 = \|1 - p\|_2 \leq \varepsilon_2. \tag{6.8}$$

From (6.1) and (6.8) we obtain the estimate

$$\begin{aligned}\|1 - bsu^*w\|_2 = \|uw^* - ubsu^*\|_2 &\leq \|uw^* - v\|_2 + \|v - uau^*\|_2 \\ &\leq \varepsilon_1 + \varepsilon_2.\end{aligned} \tag{6.9}$$

Let $c = \mathbb{E}_{\mathcal{A}}(su^*w) \in \mathcal{A}$, $\|c\| \leq 1$, and apply $\mathbb{E}_{\mathcal{A}}$ to (6.9) to obtain

$$\|1 - bc\|_2 \leq \varepsilon_1 + \varepsilon_2. \tag{6.10}$$

For each $\omega \in \Omega$,

$$|\, 1 - b(\omega)c(\omega)\, | \geq |\, 1 - |b(\omega)c(\omega)|\, | \geq 1 - b(\omega), \tag{6.11}$$

from which it follows that

$$(1 - b)^2 \leq (1 - bc)(1 - bc)^*. \tag{6.12}$$

Apply the trace to (6.12) and use (6.10) to reach

$$\|1 - b\|_2 \leq \varepsilon_1 + \varepsilon_2. \tag{6.13}$$

Thus

$$\|a - s\|_2 = \|bs - s\|_2 = \|b - 1\|_2 \leq \varepsilon_1 + \varepsilon_2. \tag{6.14}$$

From (6.1), (6.13) and the triangle inequality,

$$\begin{aligned}\|v - usu^*\|_2 &= \|v - ubsu^* + u(b - 1)su^*\|_2 \\ &= \|v - \mathbb{E}_{u\mathcal{A}u^*}(v) + u(b - 1)su^*\|_2 \leq 2\varepsilon_1 + \varepsilon_2.\end{aligned} \tag{6.15}$$

This leads to the estimate

$$\begin{aligned}\|u-ws\|_2 &= \|su^*w-1\|_2 = \|usu^*w-u\|_2\\ &= \|usu^*w-up+u(p-1)\|_2 \le \|usu^*w-up\|_2+\varepsilon_2\\ &= \|usu^*w-vw\|_2+\varepsilon_2 = \|usu^*-v\|_2+\varepsilon_2\\ &\le 2(\varepsilon_1+\varepsilon_2), \qquad (6.16)\end{aligned}$$

using (6.7) and (6.15). Now define $\tilde{u}=ws$, which is in $N(\mathcal{A})$ since s is a unitary in $\mathcal{A}$. The last inequality gives (6.3). □

The constant 90 in the next theorem is not the best possible. An earlier version of the paper used methods more specific to masas and obtained the lower estimate 31. This may be viewed at the *Mathematics ArXiv*, OA/0111330.

Theorem 6.2. *Let $\mathcal{A}$ be a masa in a separably acting type II_1 factor $\mathcal{N}$, and let $u\in\mathcal{N}$ be a unitary. Then*

$$d(u,N(\mathcal{A})) \le 90\|(I-\mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \le 90\|\mathbb{E}_{\mathcal{A}}-\mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}. \qquad (6.17)$$

Proof. Define ε to be $\|(I-\mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}$. If $\varepsilon=0$ then $\mathbb{E}_{\mathcal{A}}=\mathbb{E}_{u\mathcal{A}u^*}$ and $u\in N(\mathcal{A})$, so there is nothing to prove. Thus assume $\varepsilon>0$. Let $\mathcal{B}=u\mathcal{A}u^*$.

By Proposition 2.4, there exists $h\in\mathcal{A}'\cap\langle\mathcal{N},\mathcal{B}\rangle$ satisfying

$$\|h-e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \le \varepsilon. \qquad (6.18)$$

Applying Lemma 4.2 (ii), the spectral projection f of h corresponding to the interval [1/2,1] lies in $\mathcal{A}'\cap\langle\mathcal{N},e_{\mathcal{B}}\rangle$ and satisfies

$$\|f-e_{\mathcal{B}}\|_{2,\mathrm{Tr}} \le 2\varepsilon, \qquad (6.19)$$

(see Corollary 2.5). Theorem 5.3 (with δ replaced by ε) gives the existence of a partial isometry $v\in\mathcal{N}$ satisfying

$$v^*v\in\mathcal{A},\quad vv^*\in\mathcal{B}=u\mathcal{A}u^*,\quad v\mathcal{A}v^*=\mathcal{B}vv^*=u\mathcal{A}u^*vv^*, \qquad (6.20)$$

$$\|v-\mathbb{E}_{u\mathcal{A}u^*}(v)\|_2 \le 30\varepsilon, \qquad (6.21)$$

$$\|v\|_2^2=\tau(vv^*)\ge 1-(15\varepsilon)^2. \qquad (6.22)$$

We may now apply Proposition 6.1, with $\varepsilon_1=30\varepsilon$ and $\varepsilon_2=15\varepsilon$, to obtain a normalizing unitary $\tilde{u}\in N(\mathcal{A})$ satisfying

$$\|u-\tilde{u}\|_2 \le 2(30+15)\varepsilon = 90\varepsilon, \qquad (6.23)$$

and this is the first inequality. The second is simply

$$\begin{aligned}\|(I-\mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} &= \|(\mathbb{E}_{\mathcal{A}}-\mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}\\ &\leq \|\mathbb{E}_{\mathcal{A}}-\mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}, \qquad (6.24)\end{aligned}$$

completing the proof. □

Lemma 6.3. *If $\mathcal{A}$ is a von Neumann subalgebra of a type* II_1 *factor $\mathcal{N}$ and $u \in \mathcal{N}$ is a unitary, then*

$$\|\mathbb{E}_{\mathcal{A}}-\mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4d(u, N(\mathcal{A})). \qquad (6.25)$$

Proof. Let $v \in N(\mathcal{A})$ and define w to be uv^*. Then $w\mathcal{A}w^* = u\mathcal{A}u^*$, so it suffices to estimate $\|\mathbb{E}_{\mathcal{A}}-\mathbb{E}_{w\mathcal{A}w^*}\|_{\infty,2}$. Let $h = 1-w$. Then, for $x \in \mathcal{N}$, $\|x\| \leq 1$,

$$\begin{aligned}\|\mathbb{E}_{\mathcal{A}}(x)-\mathbb{E}_{w\mathcal{A}w^*}(x)\|_2 &= \|\mathbb{E}_{\mathcal{A}}(x)-w\mathbb{E}_{\mathcal{A}}(w^*xw)w^*\|_2\\ &= \|w^*\mathbb{E}_{\mathcal{A}}(x)w-\mathbb{E}_{\mathcal{A}}(w^*xw)\|_2\\ &\leq \|w^*\mathbb{E}_{\mathcal{A}}(x)w-\mathbb{E}_{\mathcal{A}}(x)\|_2+\|\mathbb{E}_{\mathcal{A}}(x)-\mathbb{E}_{\mathcal{A}}(w^*xw)\|_2\\ &\leq \|\mathbb{E}_{\mathcal{A}}(x)w-w\mathbb{E}_{\mathcal{A}}(x)\|_2+\|x-w^*xw\|_2\\ &= \|\mathbb{E}_{\mathcal{A}}(x)h-h\mathbb{E}_{\mathcal{A}}(x)\|_2+\|hx-xh\|_2\\ &\leq 4\|h\|_2 = 4\|1-uv^*\|_2 = 4\|v-u\|_2. \qquad (6.26)\end{aligned}$$

Taking the infimum of the right hand side of (6.26) over all $v \in N(\mathcal{A})$ gives (6.25). □

The next theorem summarizes the previous two results.

Theorem 6.4. *Let $\mathcal{A}$ be a masa in a separably acting type* II_1 *factor $\mathcal{N}$ and let u be a unitary in $\mathcal{N}$. Then*

$$d(u, N(\mathcal{A}))/90 \leq \|(I-\mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \leq \|\mathbb{E}_{\mathcal{A}}-\mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4d(u, N(\mathcal{A})). \qquad (6.27)$$

If $\mathcal{A}$ is singular, then $\mathcal{A}$ is (1/90)-strongly singular.

Proof. The inequalities of (6.27) are proved in Theorem 6.2 and Lemma 6.3. When $\mathcal{A}$ is singular, its normalizer is contained in $\mathcal{A}$, so

$$\|u-\mathbb{E}_{\mathcal{A}}(u)\|_2 \leq d(u, N(\mathcal{A})) \qquad (6.28)$$

holds. Then

$$\|u-\mathbb{E}_{\mathcal{A}}(u)\|_2 \leq 90\|\mathbb{E}_{\mathcal{A}}-\mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}, \qquad (6.29)$$

proving α-strong singularity with $\alpha = 1/90$. □

The right hand inequality of (6.27) is similar to

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2, \tag{6.30}$$

which we obtained in [22, Prop. 2.1], so u being close to $\mathcal{A}$ implies that $\mathcal{A}$ and $u\mathcal{A}u^*$ are also close. We remarked in the introduction that there are only two ways in which $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}$ can be small, and we now make precise this assertion and justify it.

Theorem 6.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be masas in a separably acting type II_1 factor $\mathcal{N}$, and let δ_1, δ_2, $\varepsilon > 0$.*

(i) *If there are projections $p \in \mathcal{A}$, $q \in \mathcal{B}$ and a unitary $u \in \mathcal{N}$ satisfying*

$$u^*qu = p, \quad u^*q\mathcal{B}u = p\mathcal{A}, \tag{6.31}$$

$$\|u - \mathbb{E}_{\mathcal{B}}(u)\|_2 \leq \delta_1 \tag{6.32}$$

and

$$tr(p) = tr(q) \geq 1 - {\delta_2}^2, \tag{6.33}$$

then

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq 4\delta_1 + 2\delta_2. \tag{6.34}$$

(ii) *If $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq \varepsilon$, then there are projections $p \in \mathcal{A}$ and $q \in \mathcal{B}$, and a unitary $u \in \mathcal{N}$ satisfying*

$$u^*qu = p, \quad u^*q\mathcal{B}u = p\mathcal{A}, \tag{6.35}$$

$$\|u - \mathbb{E}_{\mathcal{B}}(u)\|_2 \leq 45\varepsilon \tag{6.36}$$

and

$$tr(p) = tr(q) \geq 1 - (15\varepsilon)^2. \tag{6.37}$$

Proof. (i) Let $\mathcal{C} = u^*\mathcal{B}u$. Then

$$\|\mathbb{E}_{\mathcal{B}} - \mathbb{E}_{\mathcal{C}}\|_{\infty,2} \leq 4\delta_1, \tag{6.38}$$

from (6.30). If $x \in \mathcal{N}$ with $\|x\| \leq 1$, then

$$\begin{aligned}\|\mathbb{E}_{\mathcal{C}}(x) - \mathbb{E}_{\mathcal{A}}(x)\|_2 &\leq \|(1-p)(\mathbb{E}_{\mathcal{C}} - \mathbb{E}_{\mathcal{A}})(x)\|_2 + \|p\mathbb{E}_{\mathcal{C}}(px) - \mathbb{E}_{\mathcal{A}}(px)\|_2 \\ &\leq 2\delta_2,\end{aligned} \tag{6.39}$$

since

$$p\mathcal{C} = pu^*\mathcal{B}u = u^*q\mathcal{B}u = p\mathcal{A} \tag{6.40}$$

and $p\mathbb{E}_{\mathcal{A}}(p(\cdot))$ is the projection onto $p\mathcal{A}$. Then (6.34) follows immediately from (6.38) and (6.39).

(ii) As in the proofs of Theorem 6.2 and its preceding results Proposition 2.4, Lemma 4.2 and Theorem 5.2, there is a partial isometry $v \in \mathcal{N}$ satisfying

$$p = v^*v \in \mathcal{A}, \quad q = vv^* \in \mathcal{B}, \quad v^*q\mathcal{B}v = p\mathcal{A}, \tag{6.41}$$

$$\|v - \mathbb{E}_{\mathcal{B}}(v)\|_2 \leq 30\varepsilon \tag{6.42}$$

and

$$\mathrm{tr}(p) = \mathrm{tr}(q) \geq 1 - (15\varepsilon)^2. \tag{6.43}$$

Let w be a partial isometry which implements the equivalence

$$w^*w = 1 - p, \quad ww^* = 1 - q, \tag{6.44}$$

and let $u = v + w$. Then u is a unitary in $\mathcal{N}$, since the initial and final projections of v and w are orthogonal, and

$$u^*q\mathcal{B}u = v^*q\mathcal{B}v = p\mathcal{A}. \tag{6.45}$$

Observe that

$$\|w - \mathbb{E}_{\mathcal{B}}(w)\|_2 \leq \|w\|_2 = (\mathrm{tr}(1-p))^{1/2} \leq 15\varepsilon, \tag{6.46}$$

so that the inequality

$$\|u - \mathbb{E}_{\mathcal{B}}(u)\|_2 \leq 45\varepsilon \tag{6.47}$$

follows from (6.42) and (6.46). $\square$

Remark 6.6. Recall that $\mathcal{A} \subset_\delta \mathcal{B}$ is equivalent to $\|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \leq \delta$. In [2], Christensen defined the distance between $\mathcal{A}$ and $\mathcal{B}$ to be

$$\|\mathcal{A} - \mathcal{B}\|_2 = \max\{\|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2},\ \|(I - \mathbb{E}_{\mathcal{A}})\mathbb{E}_{\mathcal{B}}\|_{\infty,2}\}. \tag{6.48}$$

This quantity is clearly bounded by $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$, and the reverse inequality

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq 3\|\mathcal{A} - \mathcal{B}\|_2 \tag{6.49}$$

follows from [22, Lemma 5.2] and the algebraic identity

$$P - Q = P(I - Q) - (I - P)Q, \tag{6.50}$$

valid for all operators P and Q. In general, for any $x \in \mathcal{N}$, $\|x\| \leq 1$,

$$\begin{aligned} \|(\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}})(x)\|_2^2 &= \langle \mathbb{E}_{\mathcal{A}}(x), (\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}})(x)\rangle - \langle \mathbb{E}_{\mathcal{B}}(x), (\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}})(x)\rangle \\ &= \langle (I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}(x), x\rangle + \langle (I - \mathbb{E}_{\mathcal{A}})\mathbb{E}_{\mathcal{B}}(x), x\rangle \\ &\leq \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} + \|(I - \mathbb{E}_{\mathcal{A}})\mathbb{E}_{\mathcal{B}}\|_{\infty,2}, \end{aligned} \tag{6.51}$$

which gives the inequality

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq (2\|\mathcal{A} - \mathcal{B}\|_2)^{1/2}. \tag{6.52}$$

Thus the two notions of distance give equivalent metrics on the space of all subalgebras of $\mathcal{N}$. □

We close with a topological result on the space of masas, in the spirit of [2, 22], which also follows from results in [18]. We include a short proof for completeness.

Corollary 6.7. *The set of singular masas in a separably acting type* II_1 *factor is closed in the* $\|\cdot\|_{\infty,2}$*-metric.*

Proof. By Theorem 6.4, it suffices to show that those masas, which satisfy (6.29) (with any fixed $\alpha > 0$ replacing 90) for all unitaries $u \in \mathcal{N}$, form a closed subset. Consider a Cauchy sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of masas satisfying (6.28), and fix a unitary $u \in \mathcal{N}$. By [2], the set of masas is closed, so there is a masa $\mathcal{A}$ such that $\lim\limits_{n\to\infty} \|\mathbb{E}_{\mathcal{A}_n} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} = 0$. Then

$$\begin{aligned} \|u - \mathbb{E}_{\mathcal{A}}(u)\|_2 &\leq \|u - \mathbb{E}_{\mathcal{A}_n}(u)\|_2 + \|\mathbb{E}_{\mathcal{A}_n}(u) - \mathbb{E}_{\mathcal{A}}(u)\|_2 \\ &\leq \alpha\|\mathbb{E}_{u\mathcal{A}_n u^*} - \mathbb{E}_{\mathcal{A}_n}\|_{\infty,2} + \|\mathbb{E}_{\mathcal{A}_n} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} \\ &\leq \alpha\|\mathbb{E}_{u\mathcal{A}_n u^*} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} + \alpha\|\mathbb{E}_{u\mathcal{A}u^*} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} + \|\mathbb{E}_{\mathcal{A}_n} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2}, \end{aligned} \tag{6.53}$$

and the result follows by letting $n \to \infty$. □

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