COHOMOLOGY FOR FINITE INDEX INCLUSIONS OF FACTORS

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Abstract

If $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of type II_1 factors of finite index on a separable Hilbert space, and if $\mathcal{N}$ has a Cartan subalgebra then we show that $H^n(\mathcal{N}, \mathcal{M}) = 0$ for $n \geq 1$. We also show that $H^n_{cb}(\mathcal{N}, \mathcal{M}) = 0$, $n \geq 1$, for an arbitrary finite index inclusion $\mathcal{N} \subseteq \mathcal{M}$ of von Neumann algebras.

1 Introduction

The continuous Hochschild cohomology groups $H^n(\mathcal{N}, \mathcal{X})$ for a von Neumann algebra $\mathcal{N}$ and a Banach $\mathcal{N}$-bimodule $\mathcal{X}$ were first studied in a series of

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papers [10], [11], [12], [15], [16] by Johnson, Kadison and Ringrose. The primary focus was on the case $\mathcal{X} = \mathcal{N}$. The Kadison-Sakai theorem on derivations, [14], [24], had established that $H^1(\mathcal{N}, \mathcal{N}) = 0$ for all von Neumann algebras, and so it was natural to pose the question of whether $H^n(\mathcal{N}, \mathcal{N}) = 0$ for all $n \geq 2$. The work of [2], [4], [5], [15] on completely bounded cohomology gave an affirmative answer in the cases of type I, II_∞ and III von Neumann algebras, as well as some classes of type II_1 von Neumann algebras. However, the general type II_1 case is still open.

In [26], [27] we were able to show that $H^n(\mathcal{N}, \mathcal{N}) = 0$, $n \geq 2$, for type II_1 algebras with a separable predual and a Cartan subalgebra (a masa whose normalizing unitary group generates $\mathcal{N}$ as a von Neumann algebra). This is a rich class of von Neumann algebras [9], but algebras do exist without this property [28]. The purpose of this paper is to extend these results (which built upon preliminary results in [3], [19]) to $H^n(\mathcal{N}, \mathcal{M})$, $n \geq 2$, where $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of type II_1 factors of finite Jones index [13], [22]. The case $n = 1$ is already covered by a more general result of Christensen [1]. It is important to consider more general modules in place of $\mathcal{N}$ itself. For example, Connes has shown that an appropriate choice of module can distinguish between injective and non-injective von Neumann algebras [8] (see also [7]). In a different direction, Kirchberg, [17], has shown that the vanishing of $H^1(\mathcal{N}, B(H))$ is equivalent to a positive solution to the similarity problem for representations of C^*-algebras.

In the second section we establish some notation and recapitulate some standard theory for the reader's convenience. We also quote a theorem from [18] which we will used repeatedly. The third section is devoted to some preliminary results. One concerns the class of maps to which the averaging technique of [6] can be applied (Theorem 3.2), while another gives a method of estimating norms in $M_n(\mathcal{M})$ in the presence of Cartan subalgebras (Theorem 3.4). These are then applied in the last section to show that $H^n(\mathcal{N}, \mathcal{M}) = 0$ when $\mathcal{N}$ has a Cartan subalgebra and $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of factors of finite index. We also show that $H^n_{cb}(\mathcal{N}, \mathcal{M}) = 0$ for a finite index inclusion of von Neumann algebras.

We refer the reader to [23], [25] for general background on cohomology, and to [26], [27] for many of the techniques which we draw on here. However, we have taken the opportunity to streamline some of the arguments and the introduction of $*$-automorphisms in Corollary 3.5 is a useful suggestion of Florin Pop.

2 Preliminaries

A bounded map $\phi\colon\ \mathcal{E}\to\mathcal{F}$ between operator spaces lifts naturally to a bounded map $\phi^{(k)}\colon\ M_k(\mathcal{E})\to M_k(\mathcal{F})$ on the $k\times k$ matrices over $\mathcal{E}$ for each $k\geq 1$ (ϕ_k is a more standard notation for this map but we reserve this for a different purpose). Then ϕ is completely bounded if the quantity

$$\|\phi\|_{cb}\equiv\sup\{\|\phi^{(k)}\|\colon\ k\geq 1\}\tag{2.1}$$

is finite. If square matrices are replaced by the spaces $\mathrm{Row}_k(\mathcal{E})$ of rows over $\mathcal{E}$ of length k, then the corresponding supremum in (2.1) defines the row bounded norm. The inequalities

$$\|\phi\|\leq\|\phi\|_r\leq\|\phi\|_{cb}\tag{2.2}$$

are immediate from the definitions, and the interplay between these three norms is crucial for the results of this paper. We will denote by $CB(\mathcal{E},\mathcal{F})$ and $RB(\mathcal{E},\mathcal{F})$ respectively the spaces of completely bounded and row bounded maps from $\mathcal{E}$ to $\mathcal{F}$.

For an inclusion $\mathcal{N}\subseteq\mathcal{M}$ of von Neumann or C^*-algebras we denote by $\mathcal{L}^n(\mathcal{N},\mathcal{M})$ the space of n-linear bounded maps $\phi\colon\ \mathcal{N}^n\to\mathcal{M}$. The coboundary operator $\partial\colon\ \mathcal{L}^n(\mathcal{N},\mathcal{M})\to\mathcal{L}^{n+1}(\mathcal{N},\mathcal{M})$ is defined by

$$\begin{aligned}\partial\phi(x_1,\ldots,x_{n+1})&=x_1\phi(x_2,\ldots,x_{n+1})\\&\quad+\sum_{i=1}^{n}(-1)^i\phi(x_1,\ldots,x_{i-1},x_ix_{i+1},x_{i+2},\ldots,x_{n+1})\\&\quad+(-1)^{n+1}\phi(x_1,\ldots,x_n)x_{n+1}\end{aligned}\tag{2.3}$$

for $x_i\in\mathcal{N}$. Then ϕ is an n-cocycle if $\partial\phi=0$, while ϕ is said to be an n-coboundary if there exists $\psi\in\mathcal{L}^{n-1}(\mathcal{N},\mathcal{M})$ such that $\phi=\partial\psi$. A short algebraic calculation shows that $\partial\partial = 0$ and so coboundaries are cocycles. For $n \geq 2$ the cohomology group $H^n(\mathcal{N}, \mathcal{M})$ is defined to be the space of n-cocycles modulo the space of n-coboundaries. For $n = 1$, $H^1(\mathcal{N}, \mathcal{M})$ is the space of derivations modulo the space of inner derivations. The coefficient space $\mathcal{M}$ could be replaced by any Banach $\mathcal{N}$-bimodule in these definitions.

We will focus on von Neumann factors $\mathcal{N}$ of type II_1 with Cartan subalgebras $\mathcal{A}$: the defining property is that $\mathcal{A}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{N}$ whose normalizing unitary group $\mathcal{U} \subseteq \mathcal{N}$ generates $\mathcal{N}$ as a von Neumann algebra. Here $\mathcal{U}$ is the set of unitaries $u \in \mathcal{N}$ such that $u\mathcal{A}u^* = \mathcal{A}$. We will also be interested in the case when $\mathcal{N}$ has finite Jones index $[\mathcal{M} : \mathcal{N}]$ in $\mathcal{M}$ [13], [22]. For such inclusions a result of Pimsner and Popa [18] to the effect that $\mathcal{M}$ is finitely generated as both a left and right $\mathcal{N}$-module will be important. Since we will use it repeatedly, we state it here.

Theorem 2.1. *[18]. Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of type II_1 factors with $[\mathcal{M} : \mathcal{N}] < \infty$. Write $[\mathcal{M} : \mathcal{N}] = n + \alpha$ (n an integer and $0 \leq \alpha < 1$), and let $E_{\mathcal{N}}$ be the trace preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Then there exist $m_1, \ldots, m_{n+1} \in \mathcal{M}$ and a projection $p \in \mathcal{N}$ of trace α with the following properties.*

(i) $\mathcal{M} = \mathcal{N}m_1 + \mathcal{N}m_2 + \cdots + \mathcal{N}m_n + \mathcal{N}pm_{n+1}$,

(ii) $E_{\mathcal{N}}(m_j m_k^*) = 0$ *for* $j \neq k$,

(iii) $E_{\mathcal{N}}(m_j m_j^*) = 1$ *for* $1 \leq j \leq n$,

(iv) $E_{\mathcal{N}}(m_{n+1} m_{n+1}^*) = p$,

(v) $\|m_j\| \leq [\mathcal{M} : \mathcal{N}]^{1/2}, 1 \leq j \leq n+1$,

(vi) $\mathcal{M} = m_1^*\mathcal{N} + m_2^*\mathcal{N} + \cdots + m_n^*\mathcal{N} + m_{n+1}^*p\mathcal{N}$.

Properties (i)–(v) are the original formulation but (vi) follows from (i) by taking adjoints. We will use both the left and right $\mathcal{N}$-module decompositions of $\mathcal{M}$ subsequently. We note for future reference that properties (ii)–(iv) ensure that the $\mathcal{N}$-coefficients of an expansion of $m \in \mathcal{M}$ by (i) are unique. For example, if $x \in \mathcal{N}$ and $xm_1 = 0$ then

$$x = E_{\mathcal{N}}(xm_1m_1^*) = 0$$

using (iii) and the $\mathcal{N}$-linearity of $E_{\mathcal{N}}$.

3 Averaging maps

In this section we extend a result from [6] on the averaging of elements in $CB(\mathcal{N},\mathcal{N})$ to a larger class of maps $\mathcal{S} \subseteq RB(\mathcal{N}, B(H))$. While we do not have a characterization of which row bounded maps lie in $\mathcal{S}$, we will be able to show that this set does contain all maps used subsequently, and this is sufficient for our purposes.

Let $n_1, \ldots, n_k \in \mathcal{N}$ be fixed elements satisfying $\sum\limits_{i=1}^{k} n_i^* n_i \leq 1$, and define $\beta\colon\ RB(\mathcal{N}, B(H)) \to RB(\mathcal{N}, B(H))$ by

$$(\beta\phi)(x) = \sum_{i=1}^{k} \phi(xn_i^*)n_i \tag{3.1}$$

for $x \in \mathcal{N}$ and $\phi \in RB(\mathcal{N}, B(H))$.

Lemma 3.1. *The map β is a contraction in the row bounded norm.*

Proof. Fix $\phi \in RB(\mathcal{N}, B(H))$ and let $\psi = \beta\phi$. If $R = (x_1, \ldots, x_j) \in \mathrm{Row}_j(\mathcal{N})$, $\|R\| = 1$, then let $\widetilde{R} \in \mathrm{Row}_{jk}(\mathcal{N})$ be the row

$$(x_1n_1^*, \ldots, x_1n_k^*, \ldots, x_jn_1^*, \ldots, x_jn_k^*).$$

Then

$$\widetilde{R}\widetilde{R}^* = \sum_{l=1}^{j}\sum_{i=1}^{k} x_l n_i^* n_i x_l^* \leq \sum_{l=1}^{j} x_l x_l^*, \tag{3.2}$$

so

$$\|\widetilde{R}\| \leq \|R\| = 1. \tag{3.3}$$

Now form the $jk \times j$ matrix

$$A = \begin{pmatrix} C & \theta & & \theta \\ \theta & C & & \vdots \\ \theta & \theta & \ddots & \theta \\ \theta & \theta & & C \end{pmatrix} \tag{3.4}$$

where θ denotes a column of k 0's and $C^* = (n_1^*, \ldots, n_k^*)$. Then $A^*A \in M_j(\mathcal{N})$ is diagonal and each diagonal entry is C^*C. Thus $\|A\| \leq 1$. A short calculation shows that

$$\psi^{(j)}(R) = \phi^{(jk)}(\widetilde{R})A, \tag{3.5}$$

and it follows from (3.3) that

$$\|\psi^{(j)}(R)\| \leq \|\phi\|_r \|\widetilde{R}\| \; \|A\| \leq \|\phi\|_r. \tag{3.6}$$

Since R was an arbitrary row of unit norm, (3.6) shows that $\|\psi\|_r \leq \|\phi\|_r$ and β is a contraction in the row bounded norm. □

In [6] the existence of a projection $\rho\colon\ CB(\mathcal{N},\mathcal{N}) \to CB(\mathcal{N},\mathcal{N})_{\mathcal{N}}$ (the subspace of right $\mathcal{N}$-modular maps) was established for any von Neumann algebra $\mathcal{N}$, and moreover ρ was the point ultraweak limit of a net of maps $\rho_\alpha\colon\ CB(\mathcal{N},\mathcal{N}) \to CB(\mathcal{N},\mathcal{N})$ where each ρ_α had the form

$$(\rho_\alpha\phi)(x) = \sum_{j=1}^{\infty} \phi(xn_{j\alpha}^*)n_{j\alpha}, \qquad x \in \mathcal{N}, \tag{3.7}$$

where $\phi \in CB(\mathcal{N},\mathcal{N})$, $n_{j\alpha} \in \mathcal{N}$, and $\sum\limits_{j=1}^{\infty} n_{j\alpha}^* n_{j\alpha} = 1$. A simple pointwise ultraweak limit argument establishes Lemma 3.1 for infinite sums, and so equation (3.7) extends the definition of ρ_α to a contraction (in the row bounded norm) of $RB(\mathcal{N}, B(H))$ to itself. While ρ and its approximating net need not be unique, we fix one such collection for the subsequent discussion. It is not clear that the net of ρ_α's on the larger space of maps converges in any topology. To remedy this, we introduce an intermediate domain defined by convergence of the net not only to a limit, but to one of a particular kind.

Specifically, we form the subset $\mathcal{S}$ of $RB(\mathcal{N}, B(H))$ defined by the following property: $\phi \in \mathcal{S}$ if there exists an operator $t \in B(H)$ such that

$$\lim_{\alpha} (\rho_\alpha \phi)(x) = tx \tag{3.8}$$

ultraweakly for $x \in \mathcal{N}$. We then let $\rho\phi$ be the point ultraweak limit of $\rho_\alpha \phi$ for $\phi \in \mathcal{S}$. Since this domain is defined abstractly, we will have to show subsequently that it contains all maps of interest to us.

We note that $RB(\mathcal{N}, B(H))$ is a $(B(H), \mathcal{N})$ – bimodule under the following left and right actions:

$$(t\phi)(x) = t\phi(x), \qquad x \in \mathcal{N}, t \in B(H), \tag{3.9}$$

$$\phi_y(x) = \phi(yx), \qquad x, y \in \mathcal{N}, \tag{3.10}$$

for $\phi \in RB(\mathcal{N}, B(H))$.

Theorem 3.2. *For any von Neumann algebra $\mathcal{N} \subseteq B(H)$*

(i) $\mathcal{S}$ is a norm closed $(B(H), \mathcal{N})$-submodule of $RB(\mathcal{N}, B(H))$ containing $CB(\mathcal{N}, \mathcal{N})$;

(ii) If $\phi \in \mathcal{S}$, $t \in B(H)$, $y \in \mathcal{N}$ then

$$\rho(t\phi) = t(\rho\phi), \qquad \rho\phi_y = (\rho\phi)_y; \tag{3.11}$$

(iii) ρ is a contraction in the row bounded norm;

(iv) If $\phi \in \mathcal{S}$ and has range in a von Neumann algebra $\mathcal{M}$ containing $\mathcal{N}$ then there exists $m \in \mathcal{M}$ such that, for $x \in \mathcal{N}$,

$$\rho\phi(x) = mx, \quad \|m\| \leq \|\phi\|_r. \tag{3.12}$$

Moreover, if $\mathcal{N} \subseteq \mathcal{M} \subseteq B(H)$ is an inclusion of type II_1 factors of finite index then

(v) $\mathcal{S}$ contains $CB(\mathcal{N}, \mathcal{M})$.

Proof. That $\mathcal{S}$ contains $CB(\mathcal{N},\mathcal{N})$ is the original version of this theorem [6]. Part (iii) follows from Lemma 3.1 which establishes the contractivity of each ρ_α by a simple limit argument. It is then easy to see that $\mathcal{S}$ is a norm closed subspace of $RB(\mathcal{N}, B(H))$.

From (3.7), each ρ_α commutes with the left and right module actions of $B(H)$ and $\mathcal{N}$ respectively, and thus so does ρ. The remaining parts of (i) and (ii) are then immediate.

If $\phi \in \mathcal{S}$ has range in $\mathcal{M} \supseteq \mathcal{N}$ then the same is true for each $\rho_\alpha\phi$, by (3.7), and also for $\rho\phi$ by taking ultraweak limits. Putting $x = 1$ in (3.8) establishes (3.12).

Now suppose that $\mathcal{N} \subseteq \mathcal{M} \subseteq B(H)$ is an inclusion of type II_1 factors with $[\mathcal{M} : \mathcal{N}] < \infty$. By Theorem 2.1 we may write

$$\mathcal{M} = m_1^*\mathcal{N} + \cdots + m_n^*\mathcal{N} + m_{n+1}^* p\mathcal{N}. \tag{3.13}$$

If $\phi \in CB(\mathcal{N},\mathcal{M})$ then define $\phi_i \in CB(\mathcal{N},\mathcal{N})$, $1 \leq i \leq n+1$, by

$$\phi_i(x) = E_{\mathcal{N}}(m_i\phi(x)), \qquad x \in \mathcal{N}. \tag{3.14}$$

Fix $x \in \mathcal{N}$. By (3.13) there exist $y_1, \ldots, y_{n+1} \in \mathcal{N}$ such that

$$\phi(x) = m_1^* y_1 + \cdots + m_n^* y_n + m_{n+1}^* p y_{n+1}. \tag{3.15}$$

Multiply (3.15) by m_i and apply $E_{\mathcal{N}}$ to obtain

$$E_{\mathcal{N}}(m_i\phi(x)) = y_i, \qquad 1 \leq i \leq n, \tag{3.16}$$

$$E_{\mathcal{N}}(m_{n+1}\phi(x)) = p y_{n+1}. \tag{3.17}$$

It follows from (3.14)–(3.17) that

$$\phi(x) = m_1^*\phi_1(x) + \cdots + m_{n+1}^*\phi_{n+1}(x), \qquad x \in \mathcal{N}. \tag{3.18}$$

By the module properties of (i), we see that $\phi \in \mathcal{S}$, completing the proof. $\square$

The notion of finite index inclusions of type II_1 factors can be extended to general inclusions in the following way [22]. An inclusion $\mathcal{N} \subseteq \mathcal{M}$ of von Neumann algebras is said to be of finite index if there exists a conditional expectation $E\colon\ \mathcal{M} \to \mathcal{N}$ and a constant $c > 0$ such that $E(x) \geq cx$

for all $x \in \mathcal{M}^+$. As noted in 1.1.2 of [22], such a conditional expectation is automatically normal. In this more general situation we may obtain a projection of $CB(\mathcal{N},\mathcal{M})$ onto the space $CB(\mathcal{N},\mathcal{M})_{\mathcal{N}}$ of completely bounded right $\mathcal{N}$-module maps.

Theorem 3.3. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a finite index inclusion of von Neumann algebras. Then there exists a contractive projection $\rho\colon\ CB(\mathcal{N},\mathcal{M}) \to CB(\mathcal{N},\mathcal{M})_{\mathcal{N}}$ which is the point ultraweak limit of maps ρ_α of the form (3.7). Moreover, ρ satisfies*

$$\rho(m\phi) = m(\rho\phi), \qquad \rho\phi_y = (\rho\phi)_y \tag{3.19}$$

for $\phi \in CB(\mathcal{N},\mathcal{M}), m \in \mathcal{M}$ and $y \in \mathcal{N}$.

Proof. By [6], there is a projection $\rho\colon\ CB(\mathcal{N},\mathcal{N}) \to CB(\mathcal{N},\mathcal{N})_{\mathcal{N}}$ which is the point ultraweak limit of maps ρ_α of the form (3.7). Each ρ_α has an obvious extension to a map of $CB(\mathcal{N},\mathcal{M})$ to itself, which we also denote by ρ_α. By compactness, we may drop to a subnet and assume that $\lim\limits_{\alpha}(\rho_\alpha\phi)(x)$ exists ultraweakly (in $\mathcal{M}$) for $\phi \in CB(\mathcal{N},\mathcal{M})$ and $x \in \mathcal{N}$. This limit then defines a contraction $\rho\colon\ CB(\mathcal{N},\mathcal{M}) \to CB(\mathcal{N},\mathcal{M})$, extending the one originally defined on $CB(\mathcal{N},\mathcal{N})$. The relations (3.19) are immediate from the definition of the ρ_α's, after taking point ultraweak limits. Each ρ_α leaves fixed every right $\mathcal{N}$-module map and the same is then true of ρ. It thus suffices to show that the range of ρ is $CB(\mathcal{N},\mathcal{M})_{\mathcal{N}}$. By hypothesis there is a constant $c > 0$ and a normal conditional expectation $E\colon\ \mathcal{M} \to \mathcal{N}$ such that $E(x) \geq cx$ for $x \in \mathcal{M}^+$. We note that E is $\mathcal{N}$-bimodular. For $\phi \in CB(\mathcal{N},\mathcal{M})$ and $x \in \mathcal{N}$, it follows that

$$\begin{aligned}
\rho_\alpha(E\phi)(x) &= \sum_{j=1}^{\infty}(E\phi)(xn_{j\alpha}^*)n_{j\alpha} \\
&= \sum_{j=1}^{\infty} E(\phi(xn_{j\alpha}^*))n_{j\alpha} \\
&= E\left(\sum_{j=1}^{\infty} \phi(xn_{j\alpha}^*)n_{j\alpha}\right) \\
&= E((\rho_\alpha\phi)(x)),
\end{aligned}$$

and taking the limit over α (once again using normality of E) gives

$$\rho(E\phi) = E(\rho\phi). \tag{3.20}$$

Now fix $\phi \in CB(\mathcal{N}, \mathcal{M})$, $n \in \mathcal{N}$, and a projection $e \in \mathcal{N}$, and define $b = (\rho\phi)(n(1-e))e$. Then define $\psi \in CB(\mathcal{N}, \mathcal{M})$ by

$$\psi(x) = b^*\phi(x), \qquad x \in \mathcal{N}. \tag{3.21}$$

Since $E\psi \in CB(\mathcal{N}, \mathcal{M})$, $\rho(E\psi)$ is a right $\mathcal{N}$-module map, and so also is $E(\rho\psi)$ by (3.20). Hence

$$\begin{aligned} E(b^*\rho\phi(x(1-e))e) &= (E\rho)(b^*\phi)(x(1-e))e \\ &= (E\rho\psi)(x(1-e))e \\ &= (E\rho\psi)(x(1-e)e) \\ &= 0, \end{aligned} \tag{3.22}$$

for $x \in \mathcal{N}$. Putting $x = n$ in (3.22) gives $E(b^*b) = 0$, and so we conclude that $b = 0$ from the inequalities

$$0 \le b^*b \le c^{-1}E(b^*b) = 0.$$

Since n and e were arbitrary, it follows that $(\rho\phi)(x(1-e))e = 0$ for $x \in \mathcal{N}$ and any projection $e \in \mathcal{N}$. Thus

$$\begin{aligned} (\rho\phi)(x)e - \rho\phi(xe) &= \rho\phi(xe + x(1-e))e - \rho\phi(xe) \\ &= \rho\phi(xe)e - \rho\phi(xe) \\ &= -\rho\phi(xe)(1-e) \\ &= 0, \end{aligned} \tag{3.23}$$

because $1-e$ is also a projection in $\mathcal{N}$. Since $\mathcal{N}$ is the norm closed span of its projections, right $\mathcal{N}$-modularity of $\rho\phi$ follows from (3.23). □

The next result provides a method of estimating norms in matrix algebras over a von Neumann algebra.

Theorem 3.4. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of type II_1 factors of finite index, and suppose that $\mathcal{N}$ has a Cartan subalgebra $\mathcal{A}$. Then there exists a constant c $(= 2[\mathcal{M} : \mathcal{N}]^2)$ such that, for $X \in M_k(\mathcal{M})$, $k \geq 1$,*

$$\|X\| \leq c \sup\{\|RX\| : \ R \in Row_k(\mathcal{A}), \ \|R\| \leq 1\}.$$

Proof. Fix $X \in M_k(\mathcal{M})$, $\|X\| = 1$. By Theorem 2.1 there exist matrices $Y_i \in M_k(\mathcal{N})$ such that

$$X = Y_1 W_1 + \cdots + Y_n W_n + Y_{n+1} P W_{n+1} \tag{3.24}$$

where W_i is the $k \times k$ diagonal matrix with m_i on the diagonal and P is the $k \times k$ diagonal matrix with p on the diagonal. By the triangle inequality, we may assume without loss of generality that $\|Y_1 W_1\| \geq 1/(n+1)$ (the case $\|Y_{n+1} P W_{n+1}\| \geq 1/(n+1)$ is similar). Since $\|m_1\| \leq [\mathcal{M} : \mathcal{N}]^{1/2}$, it follows that $\|Y_1\| \geq [\mathcal{M} : \mathcal{N}]^{-1/2}(n+1)^{-1}$.

Given $\varepsilon > 0$ we may find, by [[26], Proposition 4.1], $R \in \mathrm{Row}_k(\mathcal{A})$, $\|R\| = 1$, such that

$$\|RY_1\| \geq (1 - \varepsilon)\|Y_1\|. \tag{3.25}$$

Then multiply (3.24) on the left by R, on the right by W_1^*, and apply $E_{\mathcal{N} \otimes M_k} = E_{\mathcal{N}} \otimes I_k$ to obtain

$$RY_1 = E_{\mathcal{N} \otimes M_k}(RXW_1^*). \tag{3.26}$$

Since $E_{\mathcal{N} \otimes M_k}$ is completely positive and unital, it follows that

$$\|RY_1\| \leq \|RXW_1^*\| \leq \|RX\| \ \|W_1^*\| = \|RX\| \ \|m_1\| \leq \|RX\| [\mathcal{M} : \mathcal{N}]^{1/2}. \tag{3.27}$$

The previous estimates then combine to give

$$\begin{aligned} \|RX\| &\geq [\mathcal{M} : \mathcal{N}]^{-1/2}\|RY_1\| \geq [\mathcal{M} : \mathcal{N}]^{-1/2}(1 - \varepsilon)\|Y_1\| \\ &\geq (1 - \varepsilon)[\mathcal{M} : \mathcal{N}]^{-1}(n+1)^{-1}. \end{aligned} \tag{3.28}$$

Since $\varepsilon > 0$ was arbitrary, and $n + 1 \leq 2[\mathcal{M} : \mathcal{N}]$, the result follows from (3.28), where c may be taken to be $2[\mathcal{M} : \mathcal{N}]^2$. □

The next result appears to be very specialized, but will be needed in the next section.

Corollary 3.5. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of type II_1 factors with finite index, where $\mathcal{N}$ has a Cartan subalgebra $\mathcal{A}$. Let $\mu\colon\ \mathcal{N} \to \mathcal{M}$ be row bounded and suppose that there is a $*$-automorphism α of $\mathcal{A}$ such that*

$$a\mu(x) = \mu(\alpha(a)x), \qquad a \in \mathcal{A}, \quad x \in \mathcal{N}. \tag{3.29}$$

Then μ is completely bounded and

$$\|\mu\|_{cb} \leq (2[\mathcal{M} : \mathcal{N}]^2)\|\mu\|_r. \tag{3.30}$$

Proof. Fix $k \geq 1$, and consider $X \in M_k(\mathcal{N})$, $\|X\| = 1$. The automorphism $\alpha^{(k)}$ of $M_k(\mathcal{A})$ maps $\mathrm{Row}_k(\mathcal{A})$ isometrically onto itself, and thus, by Theorem 3.4,

$$\begin{aligned}\|\mu^{(k)}(X)\| &\leq 2[\mathcal{M} : \mathcal{N}]^2 \sup\{\|R\mu^{(k)}(X)\|\colon R \in \mathrm{Row}_k(\mathcal{A}),\ \|R\| = 1\}\\ &= 2[\mathcal{M} : \mathcal{N}]^2 \sup\{\|\mu^{(k)}(\alpha^{(k)}(R)X)\|\colon\ R \in \mathrm{Row}_k(\mathcal{A}),\ \|R\| = 1\}\\ &\leq 2[\mathcal{M} : \mathcal{N}]^2\|\mu\|_r,\end{aligned} \tag{3.31}$$

since the second supremum is calculated by applying μ to rows. Since $k \geq 1$ was arbitrary, we have established (3.30). □

4 The main result

For the first result we will assume that $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of type II_1 factors of finite index represented on the Hilbert space $L^2(\mathcal{M}, tr)$ which we assume to be separable (or equivalently, $\mathcal{N}$ has separable predual). We also assume that $\mathcal{N}$ has a Cartan subalgebra $\mathcal{A}$, whereupon we can find a hyperfinite factor $\mathcal{R}$ such that $\mathcal{A} \subseteq \mathcal{R} \subseteq \mathcal{N}$ and $\mathcal{R}' \cap \mathcal{N} = \mathbb{C}1$, [20]. Christensen, [1], has shown that $H^1(\mathcal{N}, \mathcal{M}) = 0$ for any inclusion $\mathcal{N} \subseteq \mathcal{M}$ of finite von Neumann algebras. Thus our examination of $H^n(\mathcal{N}, \mathcal{M})$ can be restricted to $n \geq 2$.

Theorem 4.1. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of type II_1 factors of finite index on a separable Hilbert space and suppose that $\mathcal{N}$ has a Cartan subalgebra $\mathcal{A}$. Then $H^n(\mathcal{N}, \mathcal{M}) = 0$ for $n \geq 2$.*

Proof. Let $\mathcal{U} \subseteq \mathcal{N}$ be the group of normalizing unitaries for $\mathcal{A}$. Then $\mathrm{Alg}(\mathcal{U}) = \mathrm{Span}(\mathcal{U})$, and the norm closure of $\mathrm{Alg}(\mathcal{U})$ is a C^*-algebra denoted by $C^*(\mathcal{U})$. Now fix $n \geq 2$. As in the proof of [[26], Theorem 5.1] it suffices to consider an $\mathcal{R}$-multimodular separately normal cocycle $\theta\colon\ \mathcal{N}^n \to \mathcal{M}$ and show that its restriction to $C^*(\mathcal{U})$ is a coboundary.

Fix $u_1, \ldots, u_{n-1} \in \mathcal{U}$, and define $\mu\colon\ \mathcal{N} \to \mathcal{M}$ by

$$\mu(x) = \theta(u_1, \ldots, u_{n-1}, x), \qquad x \in \mathcal{N}. \tag{4.1}$$

We first show that μ is completely bounded. Since μ is normal and right R-modular, it follows from [[26], Proposition 4.2] that μ is row bounded and

$$\|\mu\| \leq \|\mu\|_r \leq \sqrt{2}\|\mu\|. \tag{4.2}$$

Let β_i be the $*$-automorphism of $\mathcal{A}$ defined by

$$\beta_i(x) = u_i^* x u_i, \qquad x \in \mathcal{A}, \quad 1 \leq i \leq n-1, \tag{4.3}$$

and define

$$\alpha_j = \beta_j \beta_{j-1} \ldots \beta_2 \beta_1 \in \mathrm{Aut}(\mathcal{A}), \qquad 1 \leq j \leq n-1. \tag{4.4}$$

The $\mathcal{A}$-modularity of θ implies that

$$\begin{aligned} a\mu(x) &= a\theta(u_1, \ldots, u_{n-1}, x) \\ &= \theta(au_1, \ldots, u_{n-1}, x) \\ &= \theta(u_1\beta_1(a), u_2, \ldots, u_{n-1}, x) \\ &= \theta(u_1, \beta_1(a)u_2, \ldots, u_{n-1}, x), \end{aligned} \tag{4.5}$$

and repetition of this argument in (4.5) leads to

$$a\mu(x) = \mu(\alpha_{n-1}(a)x), \qquad x \in \mathcal{N}, \quad a \in \mathcal{A}. \tag{4.6}$$

It then follows from (4.6) and Corollary 3.5 that μ is completely bounded and

$$\|\mu\|_{cb} \leq (2[\mathcal{M} : \mathcal{N}]^2)\|\mu\|_r \leq (2\sqrt{2}[\mathcal{M} : \mathcal{N}]^2)\|\mu\|. \tag{4.7}$$

These inequalities are a consequence of (3.30) and (4.2). Thus $\mu \in \mathcal{S}$ (see Theorem 3.2).

By linearity, all maps of the form

$$x \mapsto \theta(y_1, \ldots, y_{n-1}, x) \tag{4.8}$$

for $y_i \in \mathrm{Alg}(\mathcal{U})$ lie in $\mathcal{S}$, and the same is true for $y_i \in C^*(\mathcal{U})$ since $\mathcal{S}$ is $\|\cdot\|_r$-closed and $\|\cdot\|$ and $\|\cdot\|_r$ are equivalent on these maps. The modular properties of $\mathcal{S}$ show that every map (with x as the variable) in the cocycle equation

$$\begin{aligned} y_1\theta(y_2, \ldots, y_n, x) + \sum_{i=1}^{n-1}(-1)^i\theta(y_1, \ldots, y_{i-1}, y_iy_{i+1}, y_{i+2}, \ldots, y_n, x) \\ + (-1)^n\theta(y_1, \ldots, y_{n-1}, y_nx) + (-1)^{n+1}\theta(y_1, \ldots, y_n)x = 0, \end{aligned} \tag{4.9}$$

for $y_i \in C^*(\mathcal{U})$, lies in $\mathcal{S}$ and so ρ may be applied to (4.9). By Theorem 3.2, there exists an element $\psi(y_1, \ldots, y_{n-1}) \in \mathcal{M}$ such that

$$\rho(\theta(y_1, \ldots, y_{n-1}, x)) = \psi(y_1, \ldots, y_{n-1})x, \qquad x \in \mathcal{N} \tag{4.10}$$

and the estimate

$$\|\psi(y_1, \ldots, y_{n-1})\| \leq \sqrt{2}\|y_1\| \ldots \|y_{n-1}\| \tag{4.11}$$

is immediate from (3.12) and (4.2). The $(n-1)$-linearity of ψ results from the n-linearity of θ and the linearity of ρ. Using Theorem 3.2 once more, ρ transforms (4.9) to

$$\begin{aligned} y_1\psi(y_2, \ldots, y_{n-1})x + \sum_{i=1}^{n-1}(-1)^i\psi(y_1, \ldots, y_{i-1}, y_iy_{i+1}, \ldots, y_n)x \\ + (-1)^n\psi(y_1, \ldots, y_{n-1})y_nx + (-1)^{n+1}\theta(y_1, \ldots, y_n)x = 0, \end{aligned} \tag{4.12}$$

for $y_i \in C^*(\mathcal{U}), x \in \mathcal{N}$. Setting $x = 1$ in (4.12) shows that the restriction of θ to $C^*(\mathcal{U})$ is the coboundary $\partial((-1)^n\psi)$, completing the proof. □

We recall from [25] that the completely bounded cohomology groups $H^n_{cb}(\mathcal{N}, \mathcal{M})$ are defined just as are $H^n(\mathcal{N}, \mathcal{M})$, but with the added requirement that all multilinear maps be completely bounded.

Theorem 4.2. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a finite index inclusion of von Neumann algebras. Then $H^n_{cb}(\mathcal{N}, \mathcal{M}) = 0$ for $n \geq 1$.*

Proof. This is identical to the last step in the preceding proof, using the projection ρ of Theorem 3.4. □

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