

UNITARY PERTURBATIONS OF MASAS IN TYPE II_1 FACTORS

Allan M. Sinclair

Roger R. Smith*

Department of Mathematics

Department of Mathematics

University of Edinburgh

Texas A&M University

Edinburgh, EH9 3JZ

College Station, TX 77843

SCOTLAND

U.S.A.

e-mail: allan@maths.ed.ac.uk

rsmith@math.tamu.edu

Abstract

The main result of this paper is the inequality

$$d(u, N(\mathcal{A}))/31 \leq \|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4d(u, N(\mathcal{A})),$$

where \mathcal{A} is a masa in a separably acting type II_1 factor \mathcal{N} , $u \in \mathcal{N}$ is a unitary, $N(\mathcal{A})$ is the group of normalizing unitaries, d is the distance measured in the $\|\cdot\|_2$ -norm, and $\|\cdot\|_{\infty,2}$ is a norm defined on the space of bounded maps on \mathcal{N} by

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2 : \|x\| \leq 1\}.$$

This result implies that a unitary which almost normalizes a masa must be close to a normalizing unitary. The inequality also shows that every singular masa is α -strongly singular for $\alpha = 1/31$.

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1 Introduction

This paper is a continuation of our work in [15, 16], where we introduced and studied strong singularity for maximal abelian self-adjoint subalgebras (masas) in a type II_1 factor \mathcal{N} . If $\phi: \mathcal{N} \rightarrow \mathcal{N}$ is a bounded linear map, then $\|\phi\|_{\infty,2}$ denotes the quantity

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2: \|x\| \leq 1\}, \quad (1.1)$$

and $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$ measures the distance between two masas \mathcal{A} and \mathcal{B} , where $\mathbb{E}_{\mathcal{A}}$ and $\mathbb{E}_{\mathcal{B}}$ are the associated trace preserving conditional expectations. The singular masas are those which contain their groups of unitary normalizers, [3]. Within this class, we introduced the notion of strongly singular masas, [15]. The defining inequality for strong singularity of a masa $\mathcal{A} \subseteq \mathcal{N}$ is

$$\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2 \leq \|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \quad (1.2)$$

for all unitaries $u \in \mathcal{N}$. Such an inequality implies that each normalizing unitary lies in \mathcal{A} , so (1.2) can only hold for singular masas. We may weaken (1.2) by inserting a constant $\alpha \in (0, 1]$ on the left hand side, and a masa which satisfies the modified inequality is called α -strongly singular. All the examples of singular masas in [15, 16] are strongly singular, and so it is natural to ask whether all singular masas have this property, or whether all are α -strongly singular (with $\alpha > 0$ perhaps depending on the particular singular masa). The first question remains open, but we will give a positive answer to the second for the value $\alpha = 1/31$. This will be a consequence of a much stronger inequality, valid for all masas, which is the main result of the paper (Theorem 5.3). Due to certain technical considerations, we will restrict attention throughout to type II_1 factors which have separable preduals or, equivalently, have representations on separable Hilbert spaces. If $u \in \mathcal{N}$ is unitary and $v \in N(\mathcal{A})$, then the inequality

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4\|u - v\|_2 \quad (1.3)$$

is straightforward (see Lemma 5.2). The converse, proved in Theorem 5.1, gives the existence of a unitary $v \in N(\mathcal{A})$ satisfying

$$\|u - v\|_2 \leq 31\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}. \quad (1.4)$$

This is a much deeper inequality, and most of the paper is devoted to its proof. It is surely not the case that 31 is the best possible constant in (1.4), but this number

emerges from a chain of various estimates and we have been unable to improve upon it.

Strong singularity of a masa \mathcal{A} was known, [15], to be related to the invariant $\delta(\mathcal{A})$ which had been introduced by Popa in 1983, [10]. Recently, [13], he showed that $\delta(\mathcal{A}) = 1$ for all singular masas \mathcal{A} , and so δ only takes the values 0 and 1. This result, [13, Cor. 2], may be stated as follows. If \mathcal{A} is a singular masa in a type II_1 factor \mathcal{N} and v is a partial isometry in \mathcal{N} with vv^* and v^*v in \mathcal{A} , then

$$\|vv^*\|_2 = \sup \{ \|x - \mathbb{E}_{\mathcal{A}}(x)\|_2 : x \in v\mathcal{A}v^*, \|x\| \leq 1 \}. \quad (1.5)$$

This result supports the possibility that all singular masas are strongly singular. The method of proof in [13] (and of the main technical lemma in [14]) uses the method developed by Christensen, [1], together with the pull-down identity of Φ from [9] and some work by Kadison on center-valued traces, [6]. Our main proof (Theorem 4.2) applies Christensen's technique to $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$, uses the pull-down map Φ , and also requires approximation of finite projections in $L^\infty[0, 1] \overline{\otimes} B(H)$. These are combined with a detailed handling of various inequalities involving projections and partial isometries.

There are two simple ways in which masas \mathcal{A} and \mathcal{B} in a type II_1 factor can be close in the $\|\cdot\|_{\infty, 2}$ -norm on their conditional expectations. If u is a unitary close to \mathcal{A} in $\|\cdot\|_2$ -norm and $u\mathcal{A}u^* = \mathcal{B}$, then $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty, 2}$ is small. Secondly, if there is a projection q of large trace in \mathcal{A} and \mathcal{B} with $q\mathcal{A} = q\mathcal{B}$, then again $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty, 2}$ is small. In Theorem 5.4 we show that a combination of these two methods is the only way in which \mathcal{A} can be close to \mathcal{B} in separably acting factors. In the last section, we will relate the distance between masas to inclusions $\mathcal{A} \subset_\delta \mathcal{B}$ or $\mathcal{B} \subset_\delta \mathcal{A}$ introduced in [8] and studied in [1]. This does not hold for general von Neumann subalgebras of \mathcal{N} and is a special property of masas.

The crucial estimates are contained in Theorem 4.2, Proposition 3.4 and Corollary 2.5; all other results in those sections are really parts of their proofs, broken down to manageable size. We recommend reading these three results in the order stated, referring back to ancillary lemmas and propositions as needed. Corollary 2.5 is essentially due to Christensen, [1], but without the norm inequalities which we have included. While we do not quote directly from [13, 14], we note that the calculations in these papers have served to motivate strongly the present work and to guide us through considerable complications. We thank Sorin Popa for providing us with copies of these two recent preprints prior to their publication.

2 Preliminaries

Let \mathcal{N} be a fixed but arbitrary separably acting type II_1 factor with faithful normalized normal trace tr , and let \mathcal{B} be a maximal abelian subalgebra (masa) of \mathcal{N} . The trace induces an inner product

$$\langle x, y \rangle = \text{tr}(y^*x), \quad x, y \in \mathcal{N}, \quad (2.1)$$

on \mathcal{N} . Then $L^2(\mathcal{N}, \text{tr})$ is the resulting completion with norm

$$\|x\|_2 = (\text{tr}(x^*x))^{1/2}, \quad x \in \mathcal{N}, \quad (2.2)$$

and when $x \in \mathcal{N}$ is viewed as a vector in this Hilbert space we will denote it by \hat{x} . The unique trace preserving conditional expectation $\mathbb{E}_{\mathcal{B}}$ of \mathcal{N} onto \mathcal{B} may be regarded as a projection in $B(L^2(\mathcal{N}), \text{tr})$, where we denote it by $e_{\mathcal{B}}$. Thus

$$e_{\mathcal{B}}(\hat{x}) = \widehat{\mathbb{E}_{\mathcal{B}}(x)}, \quad x \in \mathcal{N}. \quad (2.3)$$

Properties of the trace show that there is a conjugate linear isometry $J: L^2(\mathcal{N}, \text{tr}) \rightarrow L^2(\mathcal{N}, \text{tr})$ defined by

$$J(\hat{x}) = \widehat{x^*}, \quad x \in \mathcal{N}, \quad (2.4)$$

and it is standard that \mathcal{N} , viewed as an algebra of left multiplication operators on $L^2(\mathcal{N}, \text{tr})$, has commutant $J\mathcal{N}J$. The von Neumann algebra generated by \mathcal{N} and $e_{\mathcal{B}}$ is denoted by $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$, and has commutant $J\mathcal{B}J$. Thus $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ is a type I_{∞} von Neumann algebra, since its commutant is abelian, and its center is $J\mathcal{B}J$, a masa in \mathcal{N}' and thus isomorphic to $L^{\infty}[0, 1]$. The general theory of type I von Neumann algebras, [7], shows that there is a separable infinite dimensional Hilbert space H so that $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ and $L^{\infty}[0, 1] \overline{\otimes} B(H)$ are isomorphic. When appropriate, we will regard an element $x \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ as a uniformly bounded measurable $B(H)$ -valued function $x(t)$ on $[0, 1]$. Under this identification, the center $J\mathcal{B}J$ of $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ corresponds to those functions taking values in $\mathbb{C}I$. We denote by τ the unique semi-finite faithful normal trace on $B(H)$ which assigns the value 1 to each rank 1 projection.

The following lemma (see [9, 12]) summarizes some of the basic properties of $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ and $e_{\mathcal{B}}$.

Lemma 2.1. *Let \mathcal{N} be a separably acting type II_1 factor with a masa \mathcal{B} . Then*

$$(i) \quad e_{\mathcal{B}}xe_{\mathcal{B}} = e_{\mathcal{B}}\mathbb{E}_{\mathcal{B}}(x) = \mathbb{E}_{\mathcal{B}}(x)e_{\mathcal{B}}, \quad x \in \mathcal{N}; \quad (2.5)$$

$$(ii) \quad e_{\mathcal{B}}\langle \mathcal{N}, e_{\mathcal{B}} \rangle = e_{\mathcal{B}}\mathcal{N}, \quad \langle \mathcal{N}, e_{\mathcal{B}} \rangle e_{\mathcal{B}} = \mathcal{N}e_{\mathcal{B}}; \quad (2.6)$$

$$(iii) \quad \text{if } x \in \mathcal{N} \text{ and } e_{\mathcal{B}}x = 0, \text{ then } x = 0; \quad (2.7)$$

$$(iv) \quad e_{\mathcal{B}}\langle \mathcal{N}, e_{\mathcal{B}} \rangle e_{\mathcal{B}} = \mathcal{B}e_{\mathcal{B}} = e_{\mathcal{B}}\mathcal{B}; \quad (2.8)$$

(v) *there is a faithful normal semi-finite trace Tr on $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ which satisfies*

$$\text{Tr}(xe_{\mathcal{B}}y) = \text{tr}(xy), \quad x, y \in \mathcal{N}, \quad (2.9)$$

and in particular,

$$\text{Tr}(e_{\mathcal{B}}) = 1. \quad (2.10)$$

The following result will be needed subsequently. We denote by $\|x\|_{\text{Tr},2}$ the Hilbert space norm induced by Tr on the subspace of $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ consisting of elements satisfying $\text{Tr}(x^*x) < \infty$.

Lemma 2.2. *Let $\varepsilon > 0$. If $f \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ is a projection of finite trace and*

$$\|f - e_{\mathcal{B}}\|_{\text{Tr},2} \leq \varepsilon, \quad (2.11)$$

then there exists a central projection $z \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ such that zf and $ze_{\mathcal{B}}$ are equivalent projections in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Moreover, the following inequalities hold:

$$\|zf - ze_{\mathcal{B}}\|_{\text{Tr},2}, \quad \|ze_{\mathcal{B}} - e_{\mathcal{B}}\|_{\text{Tr},2}, \quad \|zf - e_{\mathcal{B}}\|_{\text{Tr},2} \leq \varepsilon. \quad (2.12)$$

Proof. From (2.8), $e_{\mathcal{B}}$ is an abelian projection in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ so, altering $e_{\mathcal{B}}(t)$ on a null set if necessary, each $e_{\mathcal{B}}(t)$ is a projection in $B(H)$ whose rank is at most 1. If $\{t \in [0,1]: e_{\mathcal{B}}(t) = 0\}$ were not a null set, then there would exist a non-zero central projection p corresponding to this set so that $e_{\mathcal{B}}p = 0$, contradicting (2.7). Thus we may assume that each $e_{\mathcal{B}}(t)$ has rank 1.

Since Tr is a faithful normal semi-finite trace on $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$, there exists a non-negative \mathbb{R} -valued measurable function $k(t)$ on $[0,1]$ such that

$$\text{Tr}(y) = \int_0^1 k(t)\tau(y(t))dt, \quad y \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle, \quad \text{Tr}(y^*y) < \infty. \quad (2.13)$$

By (2.10),

$$\text{Tr}(e_{\mathcal{B}}) = \int_0^1 k(t)\tau(e_{\mathcal{B}}(t))dt = 1, \quad (2.14)$$

and thus integration against $k(t)$ defines a probability measure μ on $[0,1]$ such that

$$\mathrm{Tr}(y) = \int_0^1 \tau(y(t))d\mu(t), \quad y \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle, \quad \mathrm{Tr}(y^*y) < \infty. \quad (2.15)$$

It follows from (2.15) that

$$\|y\|_{\mathrm{Tr},2}^2 = \int_0^1 \tau(y(t)^*y(t))d\mu(t), \quad y \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle. \quad (2.16)$$

Consider a rank 1 projection $p \in B(H)$ and a projection $q \in B(H)$ of rank $n \geq 2$. Then

$$\begin{aligned} \tau((p-q)^2) &= \tau(p+q-2pq) \\ &= \tau(p+q-2pqp) \\ &\geq \tau(p+q-2p) \\ &\geq 1, \end{aligned} \quad (2.17)$$

and the same inequality is obvious if $q = 0$. Let $G = \{t \in [0,1]: \mathrm{rank}(f(t)) \neq 1\}$. Then, from (2.11),

$$\begin{aligned} \varepsilon^2 &\geq \|f - e_{\mathcal{B}}\|_{\mathrm{Tr},2}^2 \\ &\geq \int_G \tau((f(t) - e_{\mathcal{B}}(t))^2)d\mu(t) \\ &\geq \mu(G), \end{aligned} \quad (2.18)$$

by (2.17). Let $z = \chi_{G^c} \otimes I$, a central projection in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Then the ranks of $z(t)f(t)$ and $z(t)e_{\mathcal{B}}(t)$ are simultaneously 0 or 1, and so zf and $ze_{\mathcal{B}}$ are equivalent projections in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Then

$$\begin{aligned} \|ze_{\mathcal{B}} - e_{\mathcal{B}}\|_{\mathrm{Tr},2}^2 &= \int_G \tau(e_{\mathcal{B}}(t))d\mu(t) \\ &= \mu(G) \\ &\leq \varepsilon^2, \end{aligned} \quad (2.19)$$

from (2.18), while

$$\begin{aligned} \|zf - e_{\mathcal{B}}\|_{\mathrm{Tr},2}^2 &= \int_G \tau(e_{\mathcal{B}}(t))d\mu(t) + \int_{G^c} \tau((f(t) - e_{\mathcal{B}}(t))^2)d\mu(t) \\ &\leq \|f - e_{\mathcal{B}}\|_{\mathrm{Tr},2}^2 \end{aligned} \quad (2.20)$$

since, on G ,

$$\tau(e_{\mathcal{B}}(t)) = 1 \leq \tau((f(t) - e_{\mathcal{B}}(t))^2), \quad (2.21)$$

by (2.17). Finally,

$$\|zf - ze_{\mathcal{B}}\|_{\text{Tr},2} \leq \|z\| \|f - e_{\mathcal{B}}\|_{\text{Tr},2} \leq \varepsilon, \quad (2.22)$$

completely the proof of (2.12). \square

We now recall some properties of the polar decomposition and some trace norm inequalities. These may be found in [2, 7].

Lemma 2.3. *Let \mathcal{M} be a von Neumann algebra.*

- (i) *If $w \in \mathcal{M}$ then there exists a partial isometry $v \in \mathcal{M}$, whose initial and final spaces are respectively the closures of the ranges of w^* and w , satisfying*

$$w = v(w^*w)^{1/2} = (ww^*)^{1/2}v. \quad (2.23)$$

- (ii) *Suppose that \mathcal{M} has a faithful normal semifinite trace Tr . If $x \in \mathcal{M}$, $0 \leq x \leq 1$, $\text{Tr}(x^*x) < \infty$, and f is the spectral projection of x corresponding to the interval $[1/2, 1]$, then*

$$\|e - f\|_{\text{Tr},2} \leq 2\|e - x\|_{\text{Tr},2} \quad (2.24)$$

for any projection $e \in \mathcal{M}$ of finite trace.

- (iii) *Suppose that \mathcal{M} has a faithful normal semifinite trace and let p and q be equivalent finite projections in \mathcal{M} . Then there exists a partial isometry $v \in \mathcal{M}$ and a unitary $u \in \mathcal{M}$ satisfying*

$$v^*v = p, \quad vv^* = q, \quad (2.25)$$

$$v|p - q| = |p - q|v, \quad (2.26)$$

$$|v - p|, |v - q| \leq 2^{1/2}|p - q|, \quad (2.27)$$

$$upu^* = q, \quad u|p - q| = |p - q|u, \quad (2.28)$$

$$|1 - u| \leq 2^{1/2}|p - q|. \quad (2.29)$$

The following result is essentially in [1], and is also used in [13, 14]. We reprove it here since the norm estimates that we obtain will be crucial for subsequent developments.

Proposition 2.4. *Let \mathcal{A} and \mathcal{B} be masas in a separably acting type II_1 factor \mathcal{N} , and let $\overline{K_{\mathcal{A}}^w(e_{\mathcal{B}})}$ be the weak closure of the set*

$$K_{\mathcal{A}}(e_{\mathcal{B}}) = \text{conv} \{ue_{\mathcal{B}}u^* : u \text{ is a unitary in } \mathcal{A}\} \quad (2.30)$$

in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Then $\overline{K}_{\mathcal{A}}^w(e_{\mathcal{B}})$ contains a unique element h of minimal $\|\cdot\|_{\text{Tr},2}$ -norm, and this element satisfies

$$(i) \quad h \in \mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle, \quad 0 \leq h \leq 1; \quad (2.31)$$

$$(ii) \quad 1 - \text{Tr}(e_{\mathcal{B}}h) \leq \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}^2; \quad (2.32)$$

$$(iii) \quad \text{Tr}(e_{\mathcal{B}}h) = \text{Tr}(h^2); \quad (2.33)$$

$$(iv) \quad \|h - e_{\mathcal{B}}\|_{\text{Tr},2} \leq \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}. \quad (2.34)$$

Proof. Each $x \in K_{\mathcal{A}}(e_{\mathcal{B}})$ satisfies $0 \leq x \leq 1$, and so the same is true for elements of $\overline{K}_{\mathcal{A}}^w(e_{\mathcal{B}})$. Moreover, each $x \in K_{\mathcal{A}}(e_{\mathcal{B}})$ has unit trace by Lemma 2.1 (v). Let P be the set of finite trace projections in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Then, for $x \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$, $x \geq 0$,

$$\text{Tr}(x) = \sup\{\text{Tr}(xp) : p \in P\}. \quad (2.35)$$

If $p \in P$ and (x_{α}) is a net in $K_{\mathcal{A}}(e_{\mathcal{B}})$ converging weakly to $x \in \overline{K}_{\mathcal{A}}^w(e_{\mathcal{B}})$, then

$$\lim_{\alpha} \text{Tr}(x_{\alpha}p) = \text{Tr}(xp), \quad (2.36)$$

and it follows from (2.35) that $\text{Tr}(x) \leq 1$. Since $x^2 \leq x$, it follows that $x \in L^2(\langle \mathcal{N}, e_{\mathcal{B}} \rangle, \text{Tr})$. Since $\text{span}\{P\}$ is norm dense in $L^2(\langle \mathcal{N}, e_{\mathcal{B}} \rangle, \text{Tr})$, we conclude from (2.36) that (x_{α}) converges weakly to x in the Hilbert space. Thus $\overline{K}_{\mathcal{A}}^w(e_{\mathcal{B}})$ is weakly compact in both $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ and $L^2(\langle \mathcal{N}, e_{\mathcal{B}} \rangle, \text{Tr})$, and so norm closed in the latter. Thus there is a unique element $h \in \overline{K}_{\mathcal{A}}^w(e_{\mathcal{B}})$ of minimal $\|\cdot\|_{\text{Tr},2}$ -norm.

For each unitary $u \in \mathcal{A}$, the map $x \mapsto uxu^*$ is a $\|\cdot\|_{\text{Tr},2}$ -norm isometry which leaves $\overline{K}_{\mathcal{A}}^w(e_{\mathcal{B}})$ invariant. Thus

$$uhu^* = h, \quad u \text{ unitary in } \mathcal{A}, \quad (2.37)$$

by minimality of h , so $h \in \mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$. This proves (i).

Consider a unitary $u \in \mathcal{A}$. Then, by Lemma 2.1,

$$\begin{aligned} 1 - \text{Tr}(e_{\mathcal{B}}ue_{\mathcal{B}}u^*) &= 1 - \text{Tr}(e_{\mathcal{B}}\mathbb{E}_{\mathcal{B}}(u)u^*) \\ &= 1 - \text{tr}(\mathbb{E}_{\mathcal{B}}(u)u^*) \\ &= 1 - \text{tr}(\mathbb{E}_{\mathcal{B}}(u)\mathbb{E}_{\mathcal{B}}(u)^*) \\ &= 1 - \|\mathbb{E}_{\mathcal{B}}(u)\|_2^2 \\ &= \|(I - \mathbb{E}_{\mathcal{B}})(u)\|_2^2 \\ &\leq \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}^2. \end{aligned} \quad (2.38)$$

This inequality persists when $ue_{\mathcal{B}}u^*$ is replaced by elements of $K_{\mathcal{A}}(e_{\mathcal{B}})$, so it follows from (2.36) that

$$1 - \text{Tr}(e_{\mathcal{B}}h) \leq \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}^2, \quad (2.39)$$

proving (ii).

Since $h \in \mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$,

$$\text{Tr}(ue_{\mathcal{B}}u^*h) = \text{Tr}(e_{\mathcal{B}}u^*hu) = \text{Tr}(e_{\mathcal{B}}h) \quad (2.40)$$

for all unitaries $u \in \mathcal{A}$. Part (iii) follows from this by taking suitable convex combinations and a weak limit to replace $ue_{\mathcal{B}}u^*$ by h on the left hand side of (2.40). Finally, using (2.39) and (2.40),

$$\begin{aligned} \|h - e_{\mathcal{B}}\|_{\text{Tr},2}^2 &= \text{Tr}(h^2 - 2he_{\mathcal{B}} + e_{\mathcal{B}}) \\ &= \text{Tr}(e_{\mathcal{B}} - he_{\mathcal{B}}) \\ &= 1 - \text{Tr}(he_{\mathcal{B}}) \\ &\leq \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}^2 \end{aligned} \quad (2.41)$$

proving (iv). □

For the last result of this section, h is the element constructed in the previous proposition.

Corollary 2.5. *Let f be the spectral projection of h corresponding to the interval $[1/2, 1]$. Then $f \in \mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$, and*

$$\|e_{\mathcal{B}} - f\|_{\text{Tr},2} \leq 2\|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}. \quad (2.42)$$

Proof. The first assertion is a consequence of elementary spectral theory. The second follows from Proposition 2.4 (iv) and Lemma 2.3 (ii). □

3 Estimates in the $\|\cdot\|_2$ -norm

This section establishes some more technical results which will be needed subsequently, the most important of which is Proposition 3.4. We maintain the conventions of the previous section: \mathcal{N} is a separably acting type II_1 factor with masas \mathcal{A} and \mathcal{B} , unless stated to the contrary.

Lemma 3.1. *Let $w \in \mathcal{N}$ have polar decomposition $w = vk$, where $k = (w^*w)^{1/2}$, and let $p = v^*v$ and $q = vv^*$ be the initial and final projections of v . If $e \in \mathcal{N}$ is a projection satisfying $ew = w$, then*

$$(i) \quad \|p - k\|_2 \leq \|e - w\|_2; \quad (3.1)$$

$$(ii) \quad \|e - q\|_2 \leq \|e - w\|_2; \quad (3.2)$$

$$(iii) \quad \|e - v\|_2 \leq 2\|e - w\|_2. \quad (3.3)$$

Proof. The first inequality is equivalent to

$$\operatorname{tr}(p + k^2 - 2pk) \leq \operatorname{tr}(e + w^*w - w - w^*), \quad (3.4)$$

since $ew = w$, and (3.4) is in turn equivalent to

$$\operatorname{tr}(w + w^*) \leq \operatorname{tr}(e - p + 2k), \quad (3.5)$$

since $pk = k$ from properties of the polar decomposition. The map $x \mapsto \operatorname{tr}(k^{1/2}xk^{1/2})$ is a positive linear functional whose norm is $\operatorname{tr}(k)$. Thus

$$|\operatorname{tr}(w)| = |\operatorname{tr}(w^*)| = |\operatorname{tr}(vk)| = |\operatorname{tr}(k^{1/2}vk^{1/2})| \leq \operatorname{tr}(k). \quad (3.6)$$

The range of e contains the range of w , so $e \geq q$. Thus

$$\operatorname{tr}(e) \geq \operatorname{tr}(q) = \operatorname{tr}(p), \quad (3.7)$$

and so (3.5) follows from (3.6), establishing (i).

The second inequality is equivalent to

$$\begin{aligned} \operatorname{tr}(q - 2eq) &\leq \operatorname{tr}(w^*w - ew - w^*e) \\ &= \operatorname{tr}(k^2 - w - w^*). \end{aligned} \quad (3.8)$$

Since $eq = q$, this is equivalent to

$$\operatorname{tr}(w + w^*) \leq \operatorname{tr}(k^2 + q) = \operatorname{tr}(p + k^2). \quad (3.9)$$

From (3.6)

$$\operatorname{tr}(w + w^*) \leq 2 \operatorname{tr}(k) = \operatorname{tr}(p + k^2 - (k - p)^2) \leq \operatorname{tr}(p + k^2), \quad (3.10)$$

which establishes (3.9) and proves (ii).

The last inequality is

$$\begin{aligned} \|e - v\|_2 &\leq \|e - vk\|_2 + \|v(k - p)\|_2 \\ &\leq \|e - w\|_2 + \|k - p\|_2 \\ &\leq 2\|e - w\|_2, \end{aligned} \quad (3.11)$$

by (i). □

The next result gives some detailed properties of the polar decomposition.

Lemma 3.2. *Let \mathcal{A} be an abelian von Neumann subalgebra of \mathcal{N} and let $\phi: \mathcal{A} \rightarrow \mathcal{N}$ be a normal $*$ -homomorphism. Let w have polar decomposition*

$$w = v(w^*w)^{1/2} = (ww^*)^{1/2}v, \quad (3.12)$$

and let $p = v^*v$, $q = vv^*$. If

$$\phi(a)w = wa, \quad a \in \mathcal{A}, \quad (3.13)$$

then

$$(i) \quad w^*w \in \mathcal{A}' \quad \text{and} \quad ww^* \in \phi(\mathcal{A})'; \quad (3.14)$$

$$(ii) \quad \phi(a)v = va \quad \text{and} \quad \phi(a)q = vav^* \quad \text{for all} \quad a \in \mathcal{A}; \quad (3.15)$$

$$(iii) \quad p \in \mathcal{A}' \cap \mathcal{N} \quad \text{and} \quad q \in \phi(\mathcal{A})' \cap \mathcal{N}. \quad (3.16)$$

Proof. If $a \in \mathcal{A}$, then

$$\begin{aligned} w^*wa &= w^*\phi(a)w = (\phi(a^*)w)^*w \\ &= (wa^*)^*w = aw^*w, \end{aligned} \quad (3.17)$$

and so $w^*w \in \mathcal{A}'$. The second statement in (i) has a similar proof.

Let f be the projection onto the closure of the range of $(w^*w)^{1/2}$. Since $w^* = (w^*w)^{1/2}v^*$, the range of w^* is contained in the range of f , and so $f \geq p$ by Lemma 2.3(i).

For all $x \in \mathcal{N}$ and $a \in \mathcal{A}$,

$$\begin{aligned}
\phi(a)v(w^*w)^{1/2}x &= \phi(a)wx \\
&= wax \\
&= v(w^*w)^{1/2}ax \\
&= va(w^*w)^{1/2}x,
\end{aligned} \tag{3.18}$$

since $(w^*w)^{1/2} \in \mathcal{A}'$ by (i). Thus

$$\phi(a)vf = vaf, \quad a \in \mathcal{A}, \tag{3.19}$$

which reduces to

$$\phi(a)v = va, \quad a \in \mathcal{A}, \tag{3.20}$$

since $f \in VN((w^*w)^{1/2}) \subseteq \mathcal{A}'$, and

$$v = vp = vpf = vf. \tag{3.21}$$

This proves the first statement in (ii). The second is immediate from

$$\phi(a)q = \phi(a)vv^* = vav^*, \quad a \in \mathcal{A}. \tag{3.22}$$

The proof of the third part is similar to that of the first, and we omit the details. \square

The next result is an approximation to show that certain projections are close to projections with similar properties, but lying in a given masa.

Proposition 3.3. *Let \mathcal{A} be an abelian von Neumann subalgebra of \mathcal{N} and let $\varepsilon > 0$. Let $q \in \mathcal{N}$ be a projection satisfying*

- (i) $\text{tr}(q) \geq 1 - \varepsilon$;
- (ii) $q \in \mathcal{A}' \cap \mathcal{N}$, and $\mathcal{A}q$ is a masa in $q\mathcal{N}q$;
- (iii) if $a \in \mathcal{A}$ and $aq = 0$, then $a = 0$.

Let \mathcal{B} be a masa containing \mathcal{A} . Then there exists a projection $\tilde{q} \in \mathcal{B}$, $\tilde{q} \leq q$, and $\text{tr}(\tilde{q}) \geq 1 - 2\varepsilon$. Moreover, we may choose \tilde{q} to have the form pq for some projection $p \in \mathcal{A}$.

Proof. We will first assume that \mathcal{A} contains the identity element of \mathcal{N} , and remove this restriction at the end. By (ii), we may define a masa in $q\mathcal{B}q$ by $\mathcal{C} = \mathcal{A}q$. By (iii), the map

$$aq \mapsto a \mapsto a(1 - q), \quad a \in \mathcal{A}, \tag{3.23}$$

is a well defined *-homomorphism θ of \mathcal{C} into $(1-q)\mathcal{N}(1-q)$. Writing elements of \mathcal{N} as matrices relative to the decomposition $1 = q + (1-q)$, we see that

$$\mathcal{A} = \left\{ \begin{pmatrix} c & 0 \\ 0 & \theta(c) \end{pmatrix} : c \in \mathcal{C} \right\}, \quad (3.24)$$

and $q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

If

$$b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathcal{B}, \quad (3.25)$$

then b commutes with \mathcal{A} , so $b_1 \in \mathcal{C}' = \mathcal{C}$, from (3.24). Applying this to b, b^* and bb^* gives $b_1, b_1^*, b_2, b_2^* \in \mathcal{C}$. Thus any partial isometry $v \in q\mathcal{B}(1-q)$ (which is then the (1,2) entry of a matrix in \mathcal{B}) has the property that $vv^* \in \mathcal{C}$.

Let $\{v_i\}_{i=1}^\infty$ be a set of partial isometries in $q\mathcal{B}(1-q)$ which is maximal with respect to having orthogonal initial projections and orthogonal final projections. Let q_1 denote the projection $\sum_{i=1}^\infty v_i v_i^* \in \mathcal{C}$. Normality of the trace implies that

$$\text{tr}(q_1) = \sum_{i=1}^\infty \text{tr}(v_i v_i^*) = \text{tr} \left(\sum_{i=1}^\infty v_i^* v_i \right) \leq \varepsilon, \quad (3.26)$$

since each projection $v_i^* v_i$ lies under $1-q$. Let $q_2 = q - q_1$, $q_3 = \theta(q_1)$, $q_4 = \theta(q_2)$. Then $\{q_i\}_{i=1}^4$ is a set of orthogonal projections which sum to 1, and we will subsequently write elements of \mathcal{N} as 4×4 matrices relative to the decomposition $1 = \sum_{i=1}^4 q_i$. Define two *-homomorphisms α and β as the restrictions of θ to $\mathcal{C}q_1$ and $\mathcal{C}q_2$ respectively. Then

$$\mathcal{A} = \left\{ \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & \alpha(f) & 0 \\ 0 & 0 & 0 & \beta(g) \end{pmatrix} : f \in \mathcal{C}q_1, g \in \mathcal{C}q_2 \right\}, \quad (3.27)$$

and $q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $q_1 + q_3 \in \mathcal{A}$ and so commutes with \mathcal{B} . This forces each $b \in \mathcal{B}$ to have the form

$$\begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix}, \quad (3.28)$$

so each element of $q\mathcal{B}(1 - q)$ has the form

$$\begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.29)$$

Since $q_1 v_i = v_i$ for all i , it follows that each v_i has the form

$$\begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.30)$$

and so

$$v_i v_i^* \leq q_1, \quad v_i^* v_i \leq q_3, \quad i \geq 1. \quad (3.31)$$

Suppose that there is an element of \mathcal{B} with a non-zero (2,4) or (4,2) entry. By taking adjoints if necessary, we may assume that the (2,4) entry is non-zero. Multiplication on the left by $q_2 + q_4 \in \mathcal{A}$ gives an element

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & g & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & y & 0 & z \end{pmatrix} \in \mathcal{B} \quad (3.32)$$

with $g \in \mathcal{C}q_2$ and $x \neq 0$. The condition $\mathcal{B} \subseteq \mathcal{A}' \cap \mathcal{N}$ leads, as before, to the conclusion that xx^* is a non-zero element of $\mathcal{C}q_2$ which, being a masa in $q_2\mathcal{N}q_2$, is isomorphic to $L^\infty[0, 1]$. We may then pick a non-negative $h \in \mathcal{C}q_2$ so that hxx^* is a non-zero projection in $\mathcal{C}q_2$. Let $k = h^{1/2}$. Then $v = kx$ is a non-zero partial isometry. Multiplication of b on the left by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta(k) \end{pmatrix} \in \mathcal{A} \quad (3.33)$$

gives an element

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & * & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \end{pmatrix} \in \mathcal{B}, \quad (3.34)$$

so

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in q\mathcal{B}(1-q) \quad (3.35)$$

is a non-zero partial isometry. The initial and final projections are under q_4 and q_2 respectively, and this contradicts the maximality of $\{v_i\}_{i=1}^\infty$. Thus no (2,4) or (4,2) entry can be non-zero, so each element of \mathcal{B} has the form

$$\begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}. \quad (3.36)$$

This implies that $q_2 \in \mathcal{B}' \cap \mathcal{N} = \mathcal{B}$. Define $p = q_2 + q_4 \in \mathcal{A}$, and let $\tilde{q} = q_2 = pq$. Then

$$\mathrm{tr}(\tilde{q}) = \mathrm{tr}(q - q_1) \geq 1 - 2\varepsilon \quad (3.37)$$

from (i) and (3.26).

We now turn to the general case. Let e be the unit of \mathcal{A} and suppose that $e \neq 1$. By (ii), $eq = q$ so $e \geq q$. Since \mathcal{A} is isomorphic to $\mathcal{A}q$, by (iii), we may define a normal $*$ -isomorphism γ from \mathcal{A} to $(1-e)\mathcal{B}(1-e)$. We may then define

$$\mathcal{A}_1 = \{a + \gamma(a) : a \in \mathcal{A}\}, \quad (3.38)$$

which also satisfies the hypotheses. From the first part, there is a projection $p \in \mathcal{A}$ such that

$$\mathrm{tr}((p + \gamma(p))q) \geq 1 - 2\varepsilon \quad (3.39)$$

and $(p + \gamma(p))q \in \mathcal{B}$. Since $\gamma(p)q = 0$, the result follows. \square

Recall from [11] that the normalizing groupoid $\mathcal{G}(\mathcal{A})$ of a masa \mathcal{A} in \mathcal{N} is the set of partial isometries $v \in \mathcal{N}$ such that $vv^*, v^*v \in \mathcal{A}$, and $v\mathcal{A}v^* = \mathcal{A}vv^*$. Such a partial isometry v implements a spatial $*$ -isomorphism between $\mathcal{A}v^*v$ and $\mathcal{A}vv^*$. By choosing a normal $*$ -isomorphism between the abelian algebras $\mathcal{A}(1-v^*v)$ and $\mathcal{A}(1-vv^*)$ (both isomorphic to $L^\infty[0,1]$), we obtain a $*$ -automorphism of \mathcal{A} satisfying the hypotheses of Lemma 2.1 of [5]. It follows that v has the form pw^* , where p is a projection in \mathcal{A} and $w \in N(\mathcal{A})$ (this result is originally in [4]). The next result will allow us to relate $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}$ to the distance from u to $N(\mathcal{A})$.

Proposition 3.4. *Let \mathcal{A} be a masa in \mathcal{N} , let $u \in \mathcal{N}$ be a unitary and let $\varepsilon_1, \varepsilon_2 > 0$. Suppose that there exists a partial isometry $v \in \mathcal{N}$ such that $v^*v \in \mathcal{A}$, $vv^* \in u\mathcal{A}u^*$, $v\mathcal{A}v^* = u\mathcal{A}u^*vv^*$, and*

$$\|v - \mathbb{E}_{u\mathcal{A}u^*}(v)\|_2 \leq \varepsilon_1, \quad (3.40)$$

$$\|v\|_2^2 \geq 1 - \varepsilon_2^2. \quad (3.41)$$

Then there exists $\tilde{u} \in N(\mathcal{A})$ such that

$$\|u - \tilde{u}\|_2 \leq 2(\varepsilon_1 + \varepsilon_2). \quad (3.42)$$

Proof. Let v_1 be the partial isometry $u^*v \in \mathcal{N}$. From the hypotheses we see that $v_1^*v_1, v_1v_1^* \in \mathcal{A}$ and

$$v_1\mathcal{A}v_1^* = u^*v\mathcal{A}v^*u = u^*u\mathcal{A}u^*vv^*u = \mathcal{A}v_1v_1^*, \quad (3.43)$$

and so $v_1 \in \mathcal{G}(\mathcal{A})$. It follows from [5] that $v_1 = pw^*$ for some projection $p \in \mathcal{A}$ and unitary $w^* \in N(\mathcal{A})$. Thus

$$vw = up. \quad (3.44)$$

From (3.40), there exists $a \in \mathcal{A}$ such that $\|a\| \leq 1$ and $\mathbb{E}_{u\mathcal{A}u^*}(v) = uau^*$. Since \mathcal{A} is abelian, it is isomorphic to $C(\Omega)$ for some compact Hausdorff space Ω . Writing $b = |a|$, $0 \leq b \leq 1$, there exists a unitary $s \in \mathcal{A}$ such that $a = bs$.

Now (3.41) and (3.44) imply that

$$\|p\|_2^2 = \|v\|_2^2 \geq 1 - \varepsilon_2^2, \quad (3.45)$$

and so

$$\|1 - p\|_2 = (1 - \|p\|_2^2)^{1/2} \leq \varepsilon_2. \quad (3.46)$$

It now follows from (3.44) that

$$\|v - uw^*\|_2 = \|vw - u\|_2 = \|up - u\|_2 = \|1 - p\|_2 \leq \varepsilon_2. \quad (3.47)$$

From (3.40) and (3.47) we obtain the estimate

$$\begin{aligned} \|1 - bsu^*w\|_2 &= \|uw^* - ub su^*\|_2 \\ &\leq \|uw^* - v\|_2 + \|v - uau^*\|_2 \\ &\leq \varepsilon_1 + \varepsilon_2. \end{aligned} \quad (3.48)$$

Let $c = \mathbb{E}_{\mathcal{A}}(su^*w) \in \mathcal{A}$, $\|c\| \leq 1$, and apply $\mathbb{E}_{\mathcal{A}}$ to (3.48) to obtain

$$\|1 - bc\|_2 \leq \varepsilon_1 + \varepsilon_2. \quad (3.49)$$

For each $\omega \in \Omega$,

$$|1 - b(\omega)c(\omega)| \geq |1 - |b(\omega)c(\omega)|| \geq 1 - b(\omega), \quad (3.50)$$

from which it follows that

$$(1 - b)^2 \leq (1 - bc)(1 - bc)^*. \quad (3.51)$$

Apply the trace to (3.51) and use (3.49) to reach

$$\|1 - b\|_2 \leq \varepsilon_1 + \varepsilon_2. \quad (3.52)$$

Thus

$$\|a - s\|_2 = \|bs - s\|_2 = \|b - 1\|_2 \leq \varepsilon_1 + \varepsilon_2. \quad (3.53)$$

From (3.40), (3.52) and the triangle inequality,

$$\begin{aligned} \|v - usu^*\|_2 &= \|v - ub su^* + u(b - 1)su^*\|_2 \\ &= \|v - \mathbb{E}_{u\mathcal{A}u^*}(v) + u(b - 1)su^*\|_2 \\ &\leq 2\varepsilon_1 + \varepsilon_2. \end{aligned} \quad (3.54)$$

This leads to the estimate

$$\begin{aligned} \|u - ws\|_2 &= \|su^*w - 1\|_2 \\ &= \|usu^*w - u\|_2 \\ &= \|usu^*w - up + u(p - 1)\|_2 \\ &\leq \|usu^*w - up\|_2 + \varepsilon_2 \\ &= \|usu^*w - vw\|_2 + \varepsilon_2 \\ &= \|usu^* - v\|_2 + \varepsilon_2 \\ &\leq 2(\varepsilon_1 + \varepsilon_2), \end{aligned} \quad (3.55)$$

using (3.46) and (3.54). Now define $\tilde{u} = ws$, which is in $N(\mathcal{A})$ since s is a unitary in \mathcal{A} . The last inequality gives (3.42). \square

4 Homomorphisms on masas

As in the previous sections, \mathcal{N} will denote a separably acting type II_1 factor with masas \mathcal{A} and \mathcal{B} . The \mathcal{N} -bimodule $\text{span}\{\mathcal{N}e_{\mathcal{B}}\mathcal{N}\}$ is weakly dense in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$, [9], and there is a well defined \mathcal{N} -bimodule map $\Phi: \text{span}\{\mathcal{N}e_{\mathcal{B}}\mathcal{N}\} \rightarrow \mathcal{N}$ which, on generators, is given by

$$\Phi(xe_{\mathcal{B}}y) = xy, \quad x, y \in \mathcal{N}. \quad (4.1)$$

For subsequent work, the importance of this map is that calculations performed in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$ can be projected down into \mathcal{N} . The first lemma collects the properties of Φ needed later (see [9, 12]).

Lemma 4.1. *The map $\Phi: \text{span}\{\mathcal{N}e_{\mathcal{B}}\mathcal{N}\} \rightarrow \mathcal{N}$ satisfies*

- (i) $\text{tr}(\Phi(x)z) = \text{Tr}(xz)$ for all $x \in \text{span}\{\mathcal{N}e_{\mathcal{B}}\mathcal{N}\}$ and for all $z \in \mathcal{N}$;
- (ii) Φ maps $e_{\mathcal{B}}\langle \mathcal{N}, e_{\mathcal{B}} \rangle = e_{\mathcal{B}}\mathcal{N}$ into \mathcal{N} and

$$\|\Phi(x)\|_2 = \|x\|_{\text{Tr},2}, \quad x \in e_{\mathcal{B}}\langle \mathcal{N}, e_{\mathcal{B}} \rangle; \quad (4.2)$$

- (iii) for all $x \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$,

$$e_{\mathcal{B}}\Phi(e_{\mathcal{B}}x) = e_{\mathcal{B}}x. \quad (4.3)$$

Proof. Consider a generator $x = se_{\mathcal{B}}t \in \mathcal{N}e_{\mathcal{B}}\mathcal{N}$. Then, for $z \in \mathcal{N}$,

$$\text{tr}(\Phi(x)z) = \text{tr}(stz) = \text{Tr}(se_{\mathcal{B}}tz) = \text{Tr}(xz). \quad (4.4)$$

The first assertion follows by taking linear combinations in (4.4).

If $x \in e_{\mathcal{B}}\mathcal{N}$ then write $x = e_{\mathcal{B}}y$ for some $y \in \mathcal{N}$. Then $\Phi(x) = y$, so

$$\|x\|_{\text{Tr},2}^2 = \text{Tr}(y^*e_{\mathcal{B}}y) = \text{tr}(y^*y) = \|\Phi(x)\|_2^2, \quad (4.5)$$

proving (4.2).

If $x \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ then $e_{\mathcal{B}}x \in e_{\mathcal{B}}\mathcal{N}$, so there exists $y \in \mathcal{N}$ so that $e_{\mathcal{B}}x = e_{\mathcal{B}}y$. Then $\Phi(e_{\mathcal{B}}x) = y$, and (4.3) is immediate. \square

The following is the main result of this section.

Theorem 4.2. *Let \mathcal{A} and \mathcal{B} be masas in \mathcal{N} , let $\varepsilon > 0$, and let f be a projection in $\mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ with*

$$\|e_{\mathcal{B}} - f\|_{\text{Tr},2} \leq \varepsilon. \quad (4.6)$$

Then there exists a partial isometry $v \in \mathcal{N}$ such that

$$v^*v \in \mathcal{A}, \quad vv^* \in \mathcal{B}, \quad vAv^* = \mathcal{B}vv^*, \quad (4.7)$$

$$\|v\|_2^2 \geq 1 - 6\varepsilon^2 \quad (4.8)$$

and

$$\|v - \mathbb{E}_{\mathcal{B}}(v)\|_2 \leq (2^{3/2} + 6^{1/2})\varepsilon. \quad (4.9)$$

Proof. Various projections will be constructed below. Those associated with \mathcal{A} will be denoted by $f, f_0, \dots, p, p_0, \dots$ and with \mathcal{B} by $e_{\mathcal{B}}, e_0, \dots, q, q_0, \dots$.

By Lemma 2.2, there is a central projection $z \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ such that $f_1 = zf \in \mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ and $e_1 = ze_{\mathcal{B}} \leq e_{\mathcal{B}}$ are equivalent projections in $\langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Moreover,

$$\begin{aligned} \|f_1 - e_{\mathcal{B}}\|_{\text{Tr},2}^2 &= \|zf - e_{\mathcal{B}}\|_{\text{Tr},2}^2 \\ &= \|z(f - e_{\mathcal{B}}) - (1 - z)e_{\mathcal{B}}\|_{\text{Tr},2}^2 \\ &= \|zf - ze_{\mathcal{B}}\|_{\text{Tr},2}^2 + \|ze_{\mathcal{B}} - e_{\mathcal{B}}\|_{\text{Tr},2}^2 \\ &\leq 2\varepsilon^2 \end{aligned} \quad (4.10)$$

by (2.12), and so

$$\|f_1 - e_{\mathcal{B}}\|_{\text{Tr},2} \leq 2^{1/2}\varepsilon. \quad (4.11)$$

Let $v_1 \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ be a partial isometry such that

$$f_1 = v_1^*v_1, \quad e_1 = v_1v_1^* \quad (4.12)$$

and

$$\|v_1 - e_1\|_{\text{Tr},2} \leq 2^{1/2}\|f_1 - e_1\|_{\text{Tr},2} \leq 2^{1/2}\varepsilon, \quad (4.13)$$

using (2.27) and (2.12).

Now define $\theta: \langle \mathcal{N}, e_{\mathcal{B}} \rangle \rightarrow e_1\langle \mathcal{N}, e_{\mathcal{B}} \rangle e_1$ by

$$\theta(x) = v_1xv_1^*, \quad x \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle. \quad (4.14)$$

Since $e_{\mathcal{B}}e_1 = e_1$ and $e_1v_1 = v_1$, the range of θ is contained in $e_{\mathcal{B}}\langle \mathcal{N}, e_{\mathcal{B}} \rangle e_{\mathcal{B}} = e_{\mathcal{B}}\mathcal{B}$, from Lemma 2.1 (iv). By Lemma 2.1 (iii) there is a well defined map $\phi: \langle \mathcal{N}, e_{\mathcal{B}} \rangle \rightarrow \mathcal{B}$, given by

$$e_{\mathcal{B}}\phi(x) = \theta(x), \quad x \in \langle \mathcal{N}, e_{\mathcal{B}} \rangle. \quad (4.15)$$

In particular, $\theta(f_1) = e_1 = e_{\mathcal{B}}e_1e_{\mathcal{B}}$, and so there is a projection $e_0 \in \mathcal{B}$ so that

$$\theta(f_1) = v_1f_1v_1^* = e_1 = e_{\mathcal{B}}e_0 = e_{\mathcal{B}}\phi(f_1). \quad (4.16)$$

Now consider $a_1, a_2 \in \mathcal{A}$, and note that $f_1 \in \mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$. Then

$$\begin{aligned} e_{\mathcal{B}}\phi(a_1a_2) &= \theta(a_1a_2) = v_1a_1a_2v_1^* \\ &= v_1f_1a_1a_2v_1^* = v_1a_1f_1a_2v_1^* \\ &= v_1a_1v_1^*v_1av_1^* = \theta(a_1)\theta(a_2) \\ &= e_{\mathcal{B}}\phi(a_1)e_{\mathcal{B}}\phi(a_2) = e_{\mathcal{B}}\phi(a_1)\phi(a_2), \end{aligned} \quad (4.17)$$

so $\phi(a_1a_2) = \phi(a_1)\phi(a_2)$ by Lemma 2.1 (iii). A similar argument shows that $\phi(a^*) = \phi(a)^*$, and so the restriction of ϕ is a normal $*$ -homomorphism of \mathcal{A} into \mathcal{B} . Since $v = v_1f_1$ and $e_{\mathcal{B}}v_1 = v_1$, we also obtain

$$\begin{aligned} \phi(a)v_1 &= \phi(a)e_{\mathcal{B}}v_1 = v_1av_1^*v_1 \\ &= v_1af_1 = v_1f_1a = v_1a \end{aligned} \quad (4.18)$$

for all $a \in \mathcal{A}$.

Now let $w = \Phi(v_1) \in \mathcal{N}$. Then it follows from Lemma 4.1 that

$$e_{\mathcal{B}}w = e_{\mathcal{B}}\Phi(v_1) = e_{\mathcal{B}}\Phi(e_{\mathcal{B}}v_1) = e_{\mathcal{B}}v_1 = v_1. \quad (4.19)$$

Thus, for $a \in \mathcal{A}$,

$$\begin{aligned} \phi(a)w &= \phi(a)\Phi(v_1) = \Phi(\phi(a)v_1) \\ &= \Phi(v_1a) = \Phi(v_1)a = wa, \end{aligned} \quad (4.20)$$

using (4.18) and the \mathcal{N} -modularity of Φ . Moreover

$$\begin{aligned} \|w - e_0\|_2 &= \|e_{\mathcal{B}}(w - e_0)\|_{\text{Tr},2} \\ &= \|v_1 - e_1\|_{\text{Tr},2} \\ &\leq 2^{1/2}\varepsilon, \end{aligned} \quad (4.21)$$

using (4.13). Now let

$$w = v_2(w^*w)^{1/2} = (ww^*)^{1/2}v_2 \quad (4.22)$$

be the polar decomposition of w (see Lemma 2.3 (i)), and let $p = v_2^*v_2$, $q = v_2v_2^*$. Equation (4.20) shows that the hypotheses of Lemma 3.2 are satisfied and so $p \in \mathcal{A}' \cap \mathcal{N}$, $q \in \phi(\mathcal{A})' \cap \mathcal{N}$, and

$$\phi(a)v_2 = v_2a, \quad \phi(a)q = v_2av_2^*, \quad a \in \mathcal{A}. \quad (4.23)$$

Since

$$\begin{aligned}
e_0 w &= e_0 \Phi(v_1) = \Phi(e_0 v_1) \\
&= \Phi(e_0 e_{\mathcal{B}} v_1) = \Phi(e_1 v_1) \\
&= \Phi(v_1) = w,
\end{aligned} \tag{4.24}$$

we may apply Lemma 3.1 to w and e_0 to obtain

$$\|e_0 - q\|_2 \leq \|e_0 - w\|_2 \leq 2^{1/2} \varepsilon, \tag{4.25}$$

where the latter inequality is (4.21). Putting $a = 1$ in (4.23) gives $e_0 q = q$, so $e_0 \geq q$.

Thus

$$\operatorname{tr}(e_0 - q) = \|e_0 - q\|_2^2 \leq 2\varepsilon^2, \tag{4.26}$$

by (4.25), and so

$$\operatorname{tr}(q) \geq \operatorname{tr}(e_0) - 2\varepsilon^2. \tag{4.27}$$

Since

$$\begin{aligned}
\operatorname{tr}(e_0) &= \operatorname{Tr}(e_{\mathcal{B}} e_0) = \operatorname{Tr}(e_1) \\
&= \operatorname{Tr}(e_{\mathcal{B}}) - \operatorname{Tr}(e_{\mathcal{B}} - e_1) \\
&= 1 - \|e_{\mathcal{B}} - e_1\|_{\operatorname{Tr}, 2}^2 \\
&\geq 1 - \varepsilon^2,
\end{aligned} \tag{4.28}$$

by (2.12) with $e_1 = z e_{\mathcal{B}}$, these two inequalities combine to give

$$\operatorname{tr}(q) \geq 1 - 3\varepsilon^2. \tag{4.29}$$

Now $\mathcal{J} = \{a \in \mathcal{A} : \phi(a) = 0\}$ is a weakly closed ideal in \mathcal{A} , so has the form $\mathcal{J} = \mathcal{A}(1 - f_2)$ for some projection $f_2 \in \mathcal{A}$. Then (4.23) shows that

$$pa = v_2^* v_2 a = v_2^* \phi(a) v_2 = 0, \quad a \in \mathcal{J}, \tag{4.30}$$

so $p(1 - f_2) = 0$. It follows that $p = p f_2$ and $f_2 \geq p$. On the other hand, (4.22) shows that $w p = w$ (since $v_2 p = v_2$), so multiplication on the right by $1 - p$ in (4.19) gives

$$v_1(1 - p) = 0, \quad e_{\mathcal{B}} \phi(1 - p) = v_1(1 - p) v_1^* = 0. \tag{4.31}$$

Thus $1 - p \in \mathcal{J}$, so $1 - p \leq 1 - f_2$. It follows that $p = f_2$ and that

$$\phi(p) = \phi(1) = e_0. \tag{4.32}$$

Now consider an element $a \in \mathcal{A}$ such that $\phi(a)q = 0$. From (4.23),

$$pa = pap = v_2^* v_2 a v_2^* v_2 = v_2^* \phi(a) q v_2 = 0 \quad (4.33)$$

and so $pa = 0$. Then $a = (1 - p)a \in \mathcal{J}$, and thus $\phi(a) = 0$. We conclude that, for $a \in \mathcal{A}$,

$$\phi(a)q = 0 \Rightarrow \phi(a) = 0. \quad (4.34)$$

Now define $\psi: p\mathcal{N}p \rightarrow q\mathcal{N}q$ by

$$\psi(x) = v_2 x v_2^*, \quad x \in p\mathcal{N}p. \quad (4.35)$$

Since $p = v_2^* v_2$ and $q = v_2 v_2^*$, it is easy to see that ψ is a normal $*$ -isomorphism. The factor $p\mathcal{N}p$ contains a masa $\mathcal{A}p$, and thus $\psi(\mathcal{A}p)$ is a masa in $q\mathcal{N}q$. From (4.32) and (4.23),

$$\psi(\mathcal{A}p) = v_2 \mathcal{A} p v_2^* = \phi(\mathcal{A}p)q = \phi(\mathcal{A})e_0 q = \phi(\mathcal{A})q, \quad (4.36)$$

so $\phi(\mathcal{A})q$ is a masa in $q\mathcal{N}q$. Now consider $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then

$$\begin{aligned} (\phi(a)q)(qbq) &= q\phi(a)bq = qb\phi(a)q \\ &= (qbq)(\phi(a)q), \end{aligned} \quad (4.37)$$

since $\phi(\mathcal{A}) \subseteq \mathcal{B}$, and $q \in \phi(\mathcal{A})'$. Thus elements of $q\mathcal{B}q$ commute with the masa $\phi(\mathcal{A})q$ in $q\mathcal{N}q$, so $q\mathcal{B}q \subseteq \phi(\mathcal{A})q$. On the other hand,

$$\phi(\mathcal{A})q = q\phi(\mathcal{A})q \subseteq q\mathcal{B}q, \quad (4.38)$$

and this establishes equality.

We have now proved that the hypotheses of Proposition 3.3 are satisfied with $\phi(\mathcal{A})$ replacing \mathcal{A} and $3\varepsilon^2$ replacing ε (see (4.29)). We conclude that there exists a projection $q_1 \in \phi(\mathcal{A})$ such that $q_1 q \in \mathcal{B}$ and

$$\text{tr}(q_1 q) \geq 1 - 6\varepsilon^2. \quad (4.39)$$

Since $\phi(\mathcal{A}) = \phi(\mathcal{A}p)$, there exists a projection $p_1 \in \mathcal{A}$, $p_1 \leq p$, so that $\phi(p_1) = q_1$.

Now define $v = v_2 p_1 \in \mathcal{N}$. Then

$$v^* v = p_1 v_2^* v_2 p_1 = p_1 p p_1 = p_1 \in \mathcal{A}, \quad (4.40)$$

and so v is a partial isometry. By (4.23),

$$v v^* = v_2 p_1 v_2^* = \phi(p_1)q = q_1 q \in \mathcal{B} \quad (4.41)$$

and thus

$$\|v\|_2^2 = \text{tr}(vv^*) = \text{tr}(q_1q) \geq 1 - 6\varepsilon^2, \quad (4.42)$$

from (4.39). If $a \in \mathcal{A}$ then

$$\begin{aligned} v\mathcal{A}v^* &= v_2p_1\mathcal{A}v_2^* = \phi(p_1\mathcal{A})q \\ &= \phi(\mathcal{A})qq_1 = \mathcal{B}vv^*, \end{aligned} \quad (4.43)$$

by (4.23) and (4.41). Finally we estimate $\|v - \mathbb{E}_{\mathcal{B}}(v)\|_2$. Since

$$\begin{aligned} \|1 - p_1\|_2^2 &= 1 - \text{tr}(p_1) \\ &\leq 1 - \text{tr}(v_2p_1v_2^*) \\ &= 1 - \|v\|_2^2 \\ &\leq 6\varepsilon^2, \end{aligned} \quad (4.44)$$

we obtain $\|1 - p_1\|_2 \leq 6^{1/2}\varepsilon$. Then

$$\begin{aligned} \|v - \mathbb{E}_{\mathcal{B}}(v)\|_2 &\leq \|v - e_0\|_2 \\ &= \|v_2p_1 - e_0\|_2 \\ &\leq \|v_2 - e_0\|_2 + \|v_2(1 - p_1)\|_2 \\ &\leq 2\|e_0 - w\|_2 + 6^{1/2}\varepsilon \\ &\leq (2^{3/2} + 6^{1/2})\varepsilon, \end{aligned} \quad (4.45)$$

where we have used (4.21) and Lemma 3.1 (iii). This completes the proof. \square

5 The main results

In the previous sections, we have completed the necessary preliminary work and we are now able to prove the main result of the paper, Theorem 5.3. This contains two inequalities which we present separately. We denote by $d(x, S)$ the distance in $\|\cdot\|_2$ -norm from an element $x \in \mathcal{N}$ to a subset $S \subseteq \mathcal{N}$.

Theorem 5.1. *Let \mathcal{A} be a masa in a separably acting type II_1 factor \mathcal{N} , and let $u \in \mathcal{N}$ be a unitary. Then*

$$d(u, N(\mathcal{A})) \leq 31\|(I - \mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \leq 31\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}. \quad (5.1)$$

Proof. Define ε to be $\|(I - \mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}$. If $\varepsilon = 0$ then $\mathbb{E}_{\mathcal{A}} = \mathbb{E}_{u\mathcal{A}u^*}$ and $u \in N(\mathcal{A})$, so there is nothing to prove. Thus assume $\varepsilon > 0$. Let $\mathcal{B} = u\mathcal{A}u^*$.

By Proposition 2.4, there exists $h \in \mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ satisfying

$$\|h - e_{\mathcal{B}}\|_{\text{Tr},2} \leq \varepsilon. \quad (5.2)$$

Applying Lemma 3.2 (ii), the spectral projection f of h corresponding to the interval $[1/2, 1]$ lies in $\mathcal{A}' \cap \langle \mathcal{N}, e_{\mathcal{B}} \rangle$ and satisfies

$$\|f - e_{\mathcal{B}}\|_{\text{Tr},2} \leq 2\varepsilon, \quad (5.3)$$

(see Corollary 2.5). Theorem 4.2 (with ε replaced by 2ε) gives the existence of a partial isometry $v \in \mathcal{N}$ satisfying

$$v^*v \in \mathcal{A}, \quad vv^* \in \mathcal{B} = u\mathcal{A}u^*, \quad v\mathcal{A}v^* = \mathcal{B}vv^* = u\mathcal{A}u^*vv^*, \quad (5.4)$$

$$\|v - \mathbb{E}_{u\mathcal{A}u^*}(v)\|_2 \leq (2^{5/2} + (24)^{1/2})\varepsilon, \quad (5.5)$$

$$\|v\|_2^2 \geq 1 - 24\varepsilon^2. \quad (5.6)$$

We may now apply Proposition 3.4, with $\varepsilon_1 = (2^{5/2} + (24)^{1/2})\varepsilon$ and $\varepsilon_2 = (24)^{1/2}\varepsilon$, to obtain a normalizing unitary $\tilde{u} \in N(\mathcal{A})$ satisfying

$$\|u - \tilde{u}\|_2 \leq 2(2^{5/2} + 2(24)^{1/2})\varepsilon. \quad (5.7)$$

Since $2(2^{5/2} + 2(24)^{1/2}) = 30.90\dots$, the first inequality follows. The second is simply

$$\begin{aligned} \|(I - \mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} &= \|(\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \\ &\leq \|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}, \end{aligned} \quad (5.8)$$

completing the proof. \square

Lemma 5.2. *If \mathcal{A} is a masa in a type II_1 factor \mathcal{N} and $u \in \mathcal{N}$ is a unitary, then*

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4d(u, N(\mathcal{A})). \quad (5.9)$$

Proof. Let $v \in N(\mathcal{A})$ and define w to be uv^* . Then $w\mathcal{A}w^* = u\mathcal{A}u^*$, so it suffices to estimate $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{w\mathcal{A}w^*}\|_{\infty,2}$. Let $h = 1 - w$. Then, for $x \in \mathcal{N}$, $\|x\| \leq 1$,

$$\begin{aligned} \|\mathbb{E}_{\mathcal{A}}(x) - \mathbb{E}_{w\mathcal{A}w^*}(x)\|_2 &= \|\mathbb{E}_{\mathcal{A}}(x) - w\mathbb{E}_{\mathcal{A}}(w^*xw)w^*\|_2 \\ &= \|w^*\mathbb{E}_{\mathcal{A}}(x)w - \mathbb{E}_{\mathcal{A}}(w^*xw)\|_2 \\ &\leq \|w^*\mathbb{E}_{\mathcal{A}}(x)w - \mathbb{E}_{\mathcal{A}}(x)\|_2 + \|\mathbb{E}_{\mathcal{A}}(x) - \mathbb{E}_{\mathcal{A}}(w^*xw)\|_2 \\ &\leq \|\mathbb{E}_{\mathcal{A}}(x)w - w\mathbb{E}_{\mathcal{A}}(x)\|_2 + \|x - w^*xw\|_2 \\ &= \|\mathbb{E}_{\mathcal{A}}(x)h - h\mathbb{E}_{\mathcal{A}}(x)\|_2 + \|hx - xh\|_2 \\ &\leq 4\|h\|_2 \\ &= 4\|1 - uv^*\|_2 \\ &= 4\|v - u\|_2. \end{aligned} \quad (5.10)$$

Taking the infimum of the right hand side of (5.10) over all $v \in N(\mathcal{A})$ gives (5.9). \square

The next theorem summarizes the previous two results.

Theorem 5.3. *Let \mathcal{A} be a masa in a separably acting type II_1 factor \mathcal{N} and let u be a unitary in \mathcal{N} . Then*

$$d(u, N(\mathcal{A}))/31 \leq \|(I - \mathbb{E}_{u\mathcal{A}u^*})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \leq \|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4d(u, N(\mathcal{A})). \quad (5.11)$$

If \mathcal{A} is singular, then \mathcal{A} is $(1/31)$ -strongly singular.

Proof. The inequalities of (5.11) are proved in Theorem 5.1 and Lemma 5.2. When \mathcal{A} is singular, its normalizer is contained in \mathcal{A} , so

$$\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2 \leq d(u, N(\mathcal{A})) \quad (5.12)$$

holds. Then

$$\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2 \leq 31\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}, \quad (5.13)$$

proving α -strong singularity with $\alpha = 1/31$. \square

The right hand inequality of (5.11) is similar to

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2} \leq 4\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2, \quad (5.14)$$

which we obtained in [15, Prop. 2.1], so u being close to \mathcal{A} implies that \mathcal{A} and $u\mathcal{A}u^*$ are also close. We remarked in the introduction that there are only two ways in which $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{u\mathcal{A}u^*}\|_{\infty,2}$ can be small, and we now make precise this assertion and justify it.

Theorem 5.4. *Let \mathcal{A} and \mathcal{B} be masas in a separably acting type II_1 factor \mathcal{N} , and let $\delta_1, \delta_2, \varepsilon > 0$.*

(i) *If there are projections $p \in \mathcal{A}$, $q \in \mathcal{B}$ and a unitary $u \in \mathcal{N}$ satisfying*

$$u^*qu = p, \quad u^*q\mathcal{B}u = p\mathcal{A}, \quad (5.15)$$

$$\|u - \mathbb{E}_{\mathcal{B}}(u)\|_2 \leq \delta_1 \quad (5.16)$$

and

$$\text{tr}(p) = \text{tr}(q) \geq 1 - \delta_2^2, \quad (5.17)$$

then

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq 4\delta_1 + 2\delta_2. \quad (5.18)$$

(ii) *If $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq \varepsilon$, then there are projections $p \in \mathcal{A}$ and $q \in \mathcal{B}$, and a unitary $u \in \mathcal{N}$ satisfying*

$$u^*qu = p, \quad u^*q\mathcal{B}u = p\mathcal{A}, \quad (5.19)$$

$$\|u - \mathbb{E}_{\mathcal{B}}(u)\|_2 \leq (2^{5/2} + 2(24)^{1/2})\varepsilon \quad (5.20)$$

and

$$\text{tr}(p) = \text{tr}(q) \geq 1 - 24\varepsilon^2. \quad (5.21)$$

Proof. (i) Let $\mathcal{C} = u^*\mathcal{B}u$. Then

$$\|\mathbb{E}_{\mathcal{B}} - \mathbb{E}_{\mathcal{C}}\|_{\infty,2} \leq 4\delta_1, \quad (5.22)$$

from (5.14). If $x \in \mathcal{N}$ with $\|x\| \leq 1$, then

$$\begin{aligned} \|\mathbb{E}_{\mathcal{C}}(x) - \mathbb{E}_{\mathcal{A}}(x)\|_2 &\leq \|(1-p)(\mathbb{E}_{\mathcal{C}} - \mathbb{E}_{\mathcal{A}})(x)\|_2 + \|p\mathbb{E}_{\mathcal{C}}(px) - \mathbb{E}_{\mathcal{A}}(px)\|_2 \\ &\leq 2\delta_2, \end{aligned} \quad (5.23)$$

since

$$p\mathcal{C} = pu^*\mathcal{B}u = u^*q\mathcal{B}u = p\mathcal{A} \quad (5.24)$$

and $p\mathbb{E}_{\mathcal{A}}(p(\cdot))$ is the projection onto $p\mathcal{A}$. Then (5.18) follows immediately from (5.22) and (5.23).

(ii) As in the proofs of Theorem 5.1 and its preceding results Proposition 2.4, Lemma 3.2 and Theorem 4.2, there is a partial isometry $v \in \mathcal{N}$ satisfying

$$p = v^*v \in \mathcal{A}, \quad q = vv^* \in \mathcal{B}, \quad v^*q\mathcal{B}v = p\mathcal{A}, \quad (5.25)$$

$$\|v - \mathbb{E}_{\mathcal{B}}(v)\|_2 \leq (2^{5/2} + (24)^{1/2})\varepsilon \quad (5.26)$$

and

$$\mathrm{tr}(p) = \mathrm{tr}(q) \geq 1 - 24\varepsilon^2. \quad (5.27)$$

Let w be a partial isometry which implements the equivalence

$$w^*w = 1 - p, \quad ww^* = 1 - q, \quad (5.28)$$

and let $u = v + w$. Then u is a unitary in \mathcal{N} , since the initial and final projections of v and w are orthogonal, and

$$u^*q\mathcal{B}u = v^*q\mathcal{B}v = p\mathcal{A}. \quad (5.29)$$

Observe that

$$\|w - \mathbb{E}_{\mathcal{B}}(w)\|_2 \leq \|w\|_2 = (\mathrm{tr}(1 - p))^{1/2} \leq (24)^{1/2}\varepsilon, \quad (5.30)$$

so that the inequality

$$\|u - \mathbb{E}_{\mathcal{B}}(u)\|_2 \leq (2^{5/2} + 2(24)^{1/2})\varepsilon \quad (5.31)$$

follows from (5.26) and (5.30). \square

Remark 5.5. Recall from [1] that Christensen defined $\mathcal{A} \subset_{\delta} \mathcal{B}$ to mean, in our terminology, that $\|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2} \leq \delta$, and then defined the distance between \mathcal{A} and \mathcal{B} to be

$$\|\mathcal{A} - \mathcal{B}\|_2 = \max \{ \|(I - \mathbb{E}_{\mathcal{B}})\mathbb{E}_{\mathcal{A}}\|_{\infty,2}, \|(I - \mathbb{E}_{\mathcal{A}})\mathbb{E}_{\mathcal{B}}\|_{\infty,2} \}. \quad (5.32)$$

This quantity is clearly bounded by $\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2}$, and the reverse inequality

$$\|\mathbb{E}_{\mathcal{A}} - \mathbb{E}_{\mathcal{B}}\|_{\infty,2} \leq 3\|\mathcal{A} - \mathcal{B}\|_2 \quad (5.33)$$

follows from [15, Lemma 5.2] and the algebraic identity

$$P - Q = P(I - Q) - (I - P)Q, \quad (5.34)$$

valid for all operators P and Q . Thus the two notions of distance give equivalent metrics on the space of masas in a type II_1 factor. \square

We close with a topological result on the space of masas, in the spirit of [1, 15], which was previously unobtainable.

Corollary 5.6. *The set of singular masas in a separably acting type II_1 factor is closed in the $\|\cdot\|_{\infty,2}$ -metric.*

Proof. By Theorem 5.3, it suffices to show that those masas, which satisfy (5.13) (with any fixed $\alpha > 0$ replacing 31) for all unitaries $u \in \mathcal{N}$, form a closed subset. Consider a Cauchy sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of masas satisfying (5.12), and fix a unitary $u \in \mathcal{N}$. By [1], the set of masas is closed, so there is a masa \mathcal{A} such that $\lim_{n \rightarrow \infty} \|\mathbb{E}_{\mathcal{A}_n} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} = 0$. Then

$$\begin{aligned}
\|u - \mathbb{E}_{\mathcal{A}}(u)\|_2 &\leq \|u - \mathbb{E}_{\mathcal{A}_n}(u)\|_2 + \|\mathbb{E}_{\mathcal{A}_n}(u) - \mathbb{E}_{\mathcal{A}}(u)\|_2 \\
&\leq \alpha \|\mathbb{E}_{u\mathcal{A}_n u^*} - \mathbb{E}_{\mathcal{A}_n}\|_{\infty,2} + \|\mathbb{E}_{\mathcal{A}_n} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} \\
&\leq \alpha \|\mathbb{E}_{u\mathcal{A}_n u^*} - \mathbb{E}_{u\mathcal{A} u^*}\|_{\infty,2} + \alpha \|\mathbb{E}_{u\mathcal{A} u^*} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} + \|\mathbb{E}_{\mathcal{A}_n} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2},
\end{aligned} \tag{5.35}$$

and the result follows by letting $n \rightarrow \infty$. □

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