Solutions #7

Q1. $$\chi_E^{-1}(\alpha, \infty) = \begin{cases} \mathbb{R} & \alpha < 0 \\ E & 0 \le \alpha < 1 \\ \phi & \alpha \geqslant 1 \end{cases}$$

Thus χ_E is measurable if and only if $E \in \mathcal{M}$.

Q2. Let $\Sigma = \{ F \subseteq \mathbb{R} : f^{-1}(F) \in \mathcal{M} \}$.

Since $f^{-1}(F^c) = f^{-1}(F)^c$, Σ is closed under complements. $f^{-1}(\mathbb{R}) = \mathbb{R}$, $f^{-1}(\phi) = \phi$, so $\mathbb{R}, \phi \in \Sigma$.

If $F_1, \ldots \in \Sigma$ then $f^{-1}\left(\bigcup_1^\infty F_i\right) = \bigcup_1^\infty f^{-1}(F_i) \in \mathcal{M}$, so $\bigcup_1^\infty F_i \in \Sigma$, and Σ is a σ-algebra.

Since f is measurable $f^{-1}(\alpha, \infty) \in \mathcal{M}$, so $(\alpha, \infty) \in \Sigma$ for all $\alpha \in \mathbb{R}$. Thus Σ contains all open intervals, so must contain all Borel sets.

Q3. Let f_n be measurable, $f_n \to f$.

$$f^{-1}[\alpha, \infty) = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geqslant N} f_n^{-1}\left(\alpha - \tfrac{1}{k}, \infty\right) \in \mathcal{M},$$ so f is measurable.

Q4 $(g\circ f)^{-1}(\alpha,\infty) = f^{-1}\left(g^{-1}(\alpha,\infty)\right).$

Since g is continuous, $g^{-1}(\alpha,\infty)$ is open, so Borel.

By Q2 $f^{-1}\left(g^{-1}(\alpha,\infty)\right) \in \mathcal{M}$, so $g\circ f$ is measurable.

Q5 Let N be a null set so that $f=g$ on N^c.

$$g^{-1}(\alpha,\infty) = \left(g^{-1}(\alpha,\infty)\cap N\right) \cup \left(g^{-1}(\alpha,\infty)\cap N^c\right)$$

$$= \left(g^{-1}(\alpha,\infty)\cap N\right) \cup \left(f^{-1}(\alpha,\infty)\cap N^c\right).$$

The first is a null set and the second is in $\mathcal{M}$

since f is measurable, so g is measurable.