WEAK CONTAINMENT RIGIDITY FOR DISTAL ACTIONS

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Abstract. We prove that if a measure distal action \( \alpha \) of a countable group \( \Gamma \) is weakly contained in a strongly ergodic probability measure preserving action \( \beta \) of \( \Gamma \), then \( \alpha \) is a factor of \( \beta \). In particular, this applies when \( \alpha \) is a compact action.

As a consequence, we show that the weak equivalence class of any strongly ergodic action completely remembers the weak isomorphism class of the maximal distal factor arising in the Furstenberg-Zimmer Structure Theorem.

1. Introduction

The notion of weak containment for group actions was introduced by A. Kechris [Ke10] as an analogue of the notion of weak containment for unitary representations. Let \( \Gamma \acts (X, \mu) \) and \( \Gamma \acts (Y, \nu) \) be two probability measure preserving (p.m.p.) actions of a countable group \( \Gamma \). Then \( \alpha \) is said to be weakly contained in \( \beta \) (in symbols, \( \alpha \prec \beta \)) if for any finite set \( S \subset \Gamma \), finite measurable partition \( \{A_i\}_{i=1}^n \) of \( X \), and \( \varepsilon > 0 \), we can find a measurable partition \( \{B_i\}_{i=1}^n \) of \( Y \) such that for all \( \gamma \in S \) and \( i,j \in \{1, \ldots, n\} \) we have

\[
|\mu(\gamma A_i \cap A_j) - \nu(\gamma B_i \cap B_j)| < \varepsilon.
\]

If \( \alpha \prec \beta \) and \( \beta \prec \alpha \), we say that \( \alpha \) is weakly equivalent to \( \beta \).

We say that \( \alpha \) is a factor of \( \beta \) if there exists a measurable, measure preserving map \( \theta : Y \to X \) such that \( \theta(\gamma y) = \gamma \theta(y) \), for all \( \gamma \in \Gamma \) and almost every \( y \in Y \). The map \( \theta \) is called a factor map from \( \beta \) to \( \alpha \). If in addition there is a conull set \( Y_0 \subseteq Y \) such that \( \theta \) is one-to-one on \( Y_0 \), then \( \theta \) is called an isomorphism and we say that \( \alpha \) is isomorphic to \( \beta \). The actions \( \alpha \) and \( \beta \) are said to be weakly isomorphic if each is a factor of the other.

As the terminology suggests, if \( \alpha \) is a factor of \( \beta \), then \( \alpha \) is weakly contained in \( \beta \). The main goal of this note is to establish a rigidity result which provides a general instance when the converse holds.

**Theorem 1.1.** Let \( \Gamma \) be a countable group, \( \Gamma \acts (X, \mu) \) be a measure distal p.m.p. action, and let \( \Gamma \acts (Y, \nu) \) be a strongly ergodic p.m.p. action. If \( \alpha \) is weakly contained in \( \beta \), then \( \alpha \) is a factor of \( \beta \).

In particular, if a compact action \( \alpha \) is weakly contained in a strongly ergodic action \( \beta \), then \( \alpha \) is a factor of \( \beta \).

Weak containment and weak equivalence have received much attention since their introduction. In [Ke10], A. Kechris shows that cost varies monotonically with weak containment and in [Ke12] Kechris uses this monotonicity to obtain a new proof that free groups have fixed price. Several other measurable combinatorial parameters of actions are known to respect weak containment and hence are invariants of weak equivalence; see [AE11, CK13, CKTD12].

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In [AW13], M. Abért and B. Weiss exhibit a remarkable anti-rigidity phenomenon for weak containment by showing that every free p.m.p. action of $\Gamma$ weakly contains the Bernoulli action over an atomless base space. The Abért-Weiss Theorem was extended in [TD15] and used to show that every weak equivalence class contains “unclassifiably many” isomorphism classes of actions, thus ruling out the possibility of weak equivalence superrigidity. These anti-rigidity results stand in marked contrast to the rigidity exhibited in Theorem 1.1 and Corollary 1.2 below, and suggest that Theorem 1.1 and Corollary 1.2 are likely optimal.

The first weak containment rigidity results were obtained by M. Abért and G. Elek. They prove that if a finite action $\alpha$ strongly ergodic action $\beta$ is a factor of $\beta$, then $\alpha$ is a factor of $\beta$ [AE10, Theorem 1]. From this, they deduce that if two strongly ergodic profinite actions are weakly equivalent, then they are isomorphic [AE10, Theorem 2]. Recall that a p.m.p. action is called profinite if it is an inverse limit of finite actions. Since any profinite action is compact, Theorem 1.1 and Corollary 1.2 below recover the results of [AE10].

Our results are new for compact non-profinite actions, and in particular for translation actions $\Gamma \curvearrowright (K, m_K)$ on connected compact groups $K$. Note that the approach of [AE10] relies on the fact that in the case of profinite actions $\Gamma \curvearrowright (X, \mu)$, there are “many” $\Gamma$-invariant measurable partitions of $X$. As such, it does not apply to translation actions on connected compact groups, since these actions admit no non-trivial invariant measurable partitions.

Instead, the proof of Theorem 1.1 relies on a new, elementary approach which we briefly outline in the case $\alpha$ is a compact action. Assuming that $\alpha \not< \beta$, we find a sequence $\theta_n : Y \to X$ of almost $\Gamma$-equivariant measurable maps. Since $\beta$ is strongly ergodic, the maps $y \mapsto d(\theta_m(y), \theta_n(y))$ must be asymptotically constant, as $m, n \to \infty$. When coupled with the compactness of the metric space $(X, d)$, this forces a subsequence of $\{\theta_n\}$ to converge almost everywhere. The limit map $\theta : Y \to X$ is $\Gamma$-equivariant, and hence realizes $\alpha$ as a factor of $\beta$.

Recall that a p.m.p. action $\Gamma \curvearrowright (Y, \nu)$ is called strongly ergodic if any sequence $A_n \subseteq Y$ of measurable sets satisfying $\nu(\gamma A_n \triangle A_n) \to 0$, for all $\gamma \in \Gamma$, is trivial, i.e. $\nu(A_n)(1 - \nu(A_n)) \to 0$. An extension $(X, \mu) \to (X_0, \mu_0)$ of ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (X_0, \mu_0)$ is called compact (or isometric) if $\Gamma \curvearrowright (X, \mu)$ is isomorphic to a homogeneous skew-product extension, i.e., if there exist a compact group $K$, a closed subgroup $L < K$, and a measurable cocycle $w : \Gamma \times X_0 \to K$, such that $\Gamma \curvearrowright (X, \mu)$ is isomorphic to $\Gamma \curvearrowright (X_0, \mu_0) \otimes (K/L, m_{K/L})$, where $m_{K/L}$ is the unique $K$-invariant Borel probability measure on $K/L$, the action is given by $\gamma(x, kL) = (\gamma x, w(\gamma, x)kL)$, and where the map $(X, \mu) \to (X_0, \mu_0)$ corresponds to the projection map $(X_0, \mu_0) \otimes (K/L, m_{K/L}) \to (X_0, \mu_0)$.

The ergodic action $\Gamma \curvearrowright (X, \mu)$ is called compact if the extension $(X, \mu) \to \{\bullet\}$, over a point, is a compact extension. Equivalently, this means that the action $\Gamma \curvearrowright (X, \mu)$ is isomorphic to an action of the form $\Gamma \curvearrowright (K/L, m_{K/L})$, where $\Gamma$ acts by translation on $K/L$ via a homomorphism $\Gamma \to K$ with dense image. In particular, in this case $X$ can be endowed with a metric $d$ such that $(X, d)$ is a compact metric space on which $\Gamma$ acts isometrically.

The distal tower associated to an ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is the directed family $(\Gamma \curvearrowright \alpha^c (X, \mu_0))_{\zeta < \omega_1}$ of factors of $\alpha$, satisfying:

1. The action $\alpha_0$ is the trivial action on a point mass;
2. The action $\alpha_{\zeta+1}$ is the largest intermediate compact extension of $\alpha_\zeta$ within $\alpha$;
3. For limit ordinals $\zeta$, the action $\alpha_\zeta$ is the inverse limit of $(\alpha_\eta)_{\eta < \zeta}$. The least countable ordinal $\eta$ for which $\alpha_{\eta+1} = \alpha_\eta$ is called the order of the tower. The action $\alpha$ is said to be (measure) distal if $\alpha = \alpha_\zeta$ for some $\zeta < \omega_1$. The Furstenberg-Zimmer Structure
Theorem [Zi76b,Fu77] states that every ergodic p.m.p. action \( \alpha \) of \( \Gamma \) has a unique maximal distal factor – namely \( \alpha_\eta \), where \( \eta \) is the order of the distal tower associated to \( \alpha \) – and that \( \alpha \) is relatively weakly mixing over this factor.

In the rest of the introduction we present several consequences of Theorem 1.1. The first consequence states that, for strongly ergodic actions, the weak isomorphism class of the maximal distal factor is an invariant of weak equivalence.

**Corollary 1.2.** Let \( \Gamma \curvearrowright (X, \mu) \) and \( \Gamma \curvearrowright (Y, \nu) \) be p.m.p. actions of a countable group \( \Gamma \). Assume that \( \beta \) is strongly ergodic. If \( \alpha \) is weakly equivalent to \( \beta \), then the maximal distal factors of \( \alpha \) and \( \beta \) are weakly isomorphic.

Moreover, if \( \beta \) is compact and strongly ergodic, then any compact action which is weakly equivalent to \( \beta \) is in fact isomorphic to \( \beta \).

**Remark.** If two compact actions are weakly isomorphic, then they are in fact isomorphic (see Lemma 2.7). This does not extend to general distal actions however; see [Le89] for an example of two distal actions of \( Z \) which are weakly isomorphic but not isomorphic.

Next, we relate Corollary 1.2 with recent spectral gap results for translation actions on connected compact groups. A p.m.p. action \( \Gamma \curvearrowright (X, \mu) \) has spectral gap if the unitary representation \( \Gamma \curvearrowright L^2(X) \otimes \mathbb{C}1 \) admits no almost invariant vectors. If an action has spectral gap, then it is strongly ergodic. Recent works of J. Bourgain and A. Gamburd [BG06,BG11], and Y. Benoist and N. de Saxc´e [BdS14] established spectral gap for a wide class of translation actions. More precisely, it was shown that if \( K \) is a simple connected compact Lie group and \( \Gamma < K \) is a countable dense subgroup generated by matrices with algebraic entries, then the translation action \( \Gamma \curvearrowright (K, m_K) \) has spectral gap (see [BdS14, Theorem 1.2]). Moreover, in this case, the left translation action \( \Gamma \curvearrowright (K/L, m_{K/L}) \) has spectral gap, for any closed subgroup \( L < K \).

This provides a large family of actions to which Corollary 1.2 applies. In particular, it allows us to construct new concrete infinite families of weakly incomparable translation actions of free groups (cf. [AE10, Theorem 3]). To this end, let \( a, b \) be integers such that \( 0 < |a| < b \) and \( \frac{a}{b} \neq \pm \frac{1}{2} \). Put \( c = b - a \) and let \( \Gamma \) be the subgroup of \( K = SO_3(\mathbb{R}) \) generated by the following rotations:

\[
A = \begin{pmatrix}
\frac{a}{b} & -\frac{\sqrt{c}}{b} & 0 \\
\frac{\sqrt{c}}{b} & \frac{a}{b} & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{a}{b} & -\frac{\sqrt{c}}{b} \\
0 & \frac{\sqrt{c}}{b} & \frac{a}{b}
\end{pmatrix}.
\]

Denote by \( \alpha_{a,b} \) the associated translation action \( \Gamma \curvearrowright^{\alpha_{(a,b)}} (K, m_K) \). For \( 2 \leq n \leq +\infty \), denote by \( \Gamma_n \) the group generated by \( \{B^kAB^{-k}|0 \leq k \leq n-1\} \), and by \( \alpha_{a,b}^n \) the restriction of \( \alpha_{a,b} \) to \( \Gamma_n \). Since \( \frac{a}{b} \neq \pm \frac{1}{2} \), [Sw94] shows that \( \Gamma \) is isomorphic to \( \mathbb{F}_2 \), and therefore \( \Gamma_n \) is isomorphic to \( \mathbb{F}_n \).

**Corollary 1.3.** Let \( (a, b), (a', b') \) be two pairs of integers as above, and \( 2 \leq n \leq +\infty \).

If \( \frac{a}{b} \neq \frac{a'}{b'} \), then \( \alpha_{(a,b)} \not\sim \alpha_{(a',b')} \) and \( \alpha_{a,b}^n \not\sim \alpha_{a',b'}^n \).

L. Bowen recently proved that if \( \beta \) is any essentially free p.m.p. action of a free group \( \Gamma = \mathbb{F}_n \), for some \( 2 \leq n \leq +\infty \), then its orbit equivalence class is weakly dense in the space of all p.m.p. actions of \( \Gamma \) (see [Bo13, Theorem 1.1]). In other words, for any p.m.p. action \( \alpha \) of \( \Gamma \), there exists a sequence \( \{\beta_n\} \) of actions of \( \Gamma \) which are orbit equivalent to \( \beta \) and converge to \( \alpha \) in the weak topology. As a consequence, \( \alpha \) is weakly contained in the infinite direct product action \( \times_{n=1}^{\infty} \beta_n \).

In view of this result, it is natural to wonder whether the sequence \( \{\beta_n\} \) can be taken constant, that is whether \( \alpha \) is weakly contained in some action which is orbit equivalent to \( \beta \). By combining
Theorem [11] with a result of I. Chifan, S. Popa and O. Sizemore [CPS11], we are able to show that this is not the case. More generally, we have:

**Corollary 1.4.** Let $\Gamma \acts\alpha (X,\mu)$ be an essentially free ergodic compact p.m.p. action of a countable non-amenable group $\Gamma$. Let $\Lambda \acts\beta (Y,\nu) = (Z,\rho)^\Lambda$ be a Bernoulli action of a countable group $\Lambda$.

Then there does not exist a p.m.p. action $\Gamma \acts\alpha (Y,\nu)$ such that

- $\alpha$ is weakly contained in $\sigma$, and
- $\sigma(\Gamma)(y) \subset \beta(\Lambda)(y)$, for almost every $y \in Y$.

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## 2. Proofs

In this section we prove the results stated in the introduction. We start with some terminology related to weak containment.

**Definition 2.1.** Let $(Y,\nu)$ be a probability space and let $X$ be a measurable space.

1. A sequence $\theta_n : Y \to X$, $n \in \mathbb{N}$, of measurable maps is said to converge weakly to the measurable map $\theta : Y \to X$, if $\nu(\theta^{-1}(A) \Delta \theta^{-1}(A)) \to 0$ for all measurable subsets $A \subseteq X$.

2. Let $\Gamma \acts\alpha (Y,\nu)$ and $\Gamma \acts\beta X$ be measurable actions of $\Gamma$ with $\alpha$ probability measure preserving. A sequence $\theta_n : Y \to X$, $n \in \mathbb{N}$, of measurable maps is said be asymptotically equivariant for the actions if $\nu(\theta_n^{-1}(\gamma^{-1} A) \Delta \gamma^{-1} \theta_n^{-1}(A)) \to 0$ for all $\gamma \in \Gamma$ and measurable subsets $A \subseteq X$.

We now have the following useful characterization of weak containment.

**Lemma 2.2.** [Ke10] Let $\Gamma \acts\alpha (X,\mu)$ and $\Gamma \acts\beta (Y,\nu)$ be p.m.p. actions.

Then $\alpha \prec \beta$ if and only if there is a sequence $\theta_n : (Y,\nu) \to (X,\mu)$, $n \in \mathbb{N}$, of measure preserving maps which are asymptotically equivariant.

**Proof.** See the proof of [Ke10] Proposition 10.1.

**Lemma 2.3.** Let $(Y,\nu)$ be a probability space, let $(X,d)$ be a Polish metric space with $d \leq 1$, and let $\mu$ a Borel probability measure on $X$.

1. Let $(\psi_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ be two sequences of measure preserving maps from $(Y,\nu)$ to $(X,\mu)$. Then $\int_Y d(\psi_n(y),\varphi_n(y)) \, d\nu \to 0$ if and only if $\nu(\psi_n^{-1}(A) \Delta \varphi_n^{-1}(A)) \to 0$ for all measurable subsets $A \subseteq X$.

2. Let $\Gamma \acts\alpha (X,\mu)$ and $\Gamma \acts\beta (Y,\nu)$ be p.m.p. actions and let $\theta_n : (Y,\nu) \to (X,\mu)$, $n \in \mathbb{N}$, be a sequence of measure preserving maps. The sequence $(\theta_n)_{n \in \mathbb{N}}$ is asymptotically equivariant if and only if $\int_Y d(\theta_n(\gamma y),\gamma \theta_n(y)) \, d\nu \to 0$ for each $\gamma \in \Gamma$. Moreover, if $(\theta_n)_{n \in \mathbb{N}}$ is asymptotically equivariant then we may find a subsequence $(\theta_{n_k})_{k \in \mathbb{N}}$ such that $d(\theta_{n_k}(\gamma y),\gamma \theta_{n_k}(y)) \to 0$, for all $\gamma \in \Gamma$ and almost every $y \in Y$.

**Proof.** (1): Assume first that $\nu(\psi_n^{-1}(A) \Delta \varphi_n^{-1}(A)) \to 0$ for all measurable subsets $A \subseteq X$. Fix $\varepsilon > 0$. Let $\{A_i\}_{i=1}^k$ be a collection of pairwise disjoint Borel subsets of $X$ such that each $A_i$ has diameter at most $\varepsilon$ and $\nu(\bigcup_{i=1}^k A_i) > 1 - \varepsilon$. Put $A_0 = X \setminus \bigcup_{i=1}^k A_i$. If $i \neq j$, then

$$\{ y \in Y | \psi_n(y) \in A_i, \varphi_n(y) \in A_j \} \subset \psi_n^{-1}(A_i) \setminus \varphi_n^{-1}(A_i).$$
Hence $\nu(\{y \in Y | \psi_n(y) \in A_i, \varphi_n(y) \in A_j\}) \to 0$. Thus, we conclude that

$$\nu\left(\bigcup_{i=0}^{k} \{y \in Y | \psi_n(y) \in A_i, \varphi_n(y) \in A_i\}\right) \to 1.$$ 

Since $\nu(\psi_n^{-1}(A_0)) = \mu(A_0) \leq \varepsilon$, we get $\liminf_n \nu\left(\bigcup_{i=1}^{k} \{y \in Y | \psi_n(y) \in A_i, \varphi_n(y) \in A_i\}\right) \geq 1 - \varepsilon$. On the other hand, if $\psi_n(y), \varphi_n(y) \in A_i$ for some $1 \leq i \leq k$, then $d(\psi_n(y), \varphi_n(y)) \leq \varepsilon$. This proves that $\liminf_n \nu(\{y \in Y | d(\psi_n(y), \varphi_n(y)) \leq \varepsilon\}) \geq 1 - \varepsilon$, for every $\varepsilon > 0$, and hence $\int_Y d(\psi_n(y), \varphi_n(y)) d\nu \to 0$.

Conversely, assume that $\int_Y d(\psi_n(y), \varphi_n(y)) d\nu \to 0$. Fix $A \subseteq X$ measurable, and $\varepsilon > 0$. Since $(X, d)$ is Polish, the measure $\mu$ is tight and regular, so we may find $L \subseteq A \subseteq U$ with $L$ compact, $U$ open, and $\mu(U \setminus L) < \varepsilon$. Letting $r = d(L, X \setminus U) \geq 0$, we have $\psi_n^{-1}(L) \setminus \varphi_n^{-1}(U) \subseteq \{y \in Y | d(\psi_n(y), \varphi_n(y)) \geq r\}$, and hence $\nu(\psi_n^{-1}(L) \setminus \varphi_n^{-1}(U)) \to 0$. Therefore, since $\psi_n$ and $\varphi_n$ are measure preserving, the containment

$$\psi_n^{-1}(A) \setminus \varphi_n^{-1}(A) \subseteq [\psi_n^{-1}(L) \setminus \varphi_n^{-1}(U)] \cup [\psi_n^{-1}(A \setminus L)] \cup [\varphi_n^{-1}(U \setminus A)]$$

implies that $\limsup_n \nu(\psi_n^{-1}(A) \setminus \varphi_n^{-1}(A)) \leq \mu(U \setminus L) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary we conclude that $\nu(\psi_n^{-1}(A) \setminus \varphi_n^{-1}(A)) \to 0$, and hence $\nu(\psi_n^{-1}(A) \Delta \varphi_n^{-1}(A)) \to 0$.

The first part of (2) is immediate from (1), and the final statement is clear. ■

The proof of Theorem 2.4 relies on the following result.

**Lemma 2.4.** Let $\Gamma \rhd^\beta (Y, \nu)$ be a strongly ergodic p.m.p. action. Let $(X, d)$ be a compact metric space, $K$ the group of isometries of $X$, and $w : \Gamma \times Y \to K$ a measurable cocycle. Assume that there exists a sequence $\theta_n : Y \to X$, $n \in \mathbb{N}$, of measurable maps such that $\int_Y d(\theta_n(\gamma y), w(\gamma, y)\theta_n(y)) d\nu(y) \to 0$, for all $\gamma \in \Gamma$.

Then there exists a measurable map $\theta : Y \to X$ such that $\theta(\gamma y) = w(\gamma, y)\theta(y)$, for all $\gamma \in \Gamma$ and almost every $y \in Y$, and there is a subsequence $(\theta_{n_k})_{k \in \mathbb{N}}$ which converges to $\theta$ both weakly and pointwise.

**Proof.** We may clearly assume $d \leq 1$. Given two measurable maps $\theta, \theta' : Y \to X$, we define

$$\tilde{d}(\theta, \theta') = \int_Y d(\theta(y), \theta'(y)) d\nu(y).$$

**Claim.** Let $\theta_n : Y \to X$ be a sequence of measurable maps and assume that for each $\gamma \in \Gamma$ we have $\int_Y d(\theta_n(\gamma y), w(\gamma, y)\theta_n(y)) d\nu(y) \to 0$. Then for every $\varepsilon > 0$, there exists $n \geq 1$ such that the set $\{m \geq n | d(\theta_m, \theta_n) < \varepsilon\}$ is infinite.

Assuming the claim, let us derive the conclusion. By using the claim we can inductively construct a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\tilde{d}(\theta_{n_{k+1}}, \theta_{n_k}) < \frac{1}{2^k}$, for any $k \geq 1$. This implies that

$$\int_Y \sum_{k=1}^{\infty} d(\theta_{n_{k+1}}(y), \theta_{n_k}(y)) d\nu(y) = \sum_{k=1}^{\infty} \tilde{d}(\theta_{n_{k+1}}, \theta_{n_k}) < 1.$$ 

Therefore, the sequence $\{\theta_{n_k}(y)\} \subseteq X$ is Cauchy and thus convergent, for almost every $y \in Y$. It is clear that the map $\theta : Y \to X$ defined as $\theta(y) := \lim_{k \to \infty} \theta_{n_k}(y)$ satisfies the conclusion.

**Proof of the claim.** Suppose that the claim is false. Thus, there exists $\varepsilon_0 > 0$ such that for all $n \geq 1$, the set $\{m \geq n | d(\theta_m, \theta_n) < \varepsilon_0\}$ is finite. Then we can inductively find a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\tilde{d}(\theta_{n_k}, \theta_{n_l}) \geq \varepsilon_0$, for all $l > k \geq 1$. 

Since $X$ is compact, it can be covered by finitely many, say $p \geq 1$, balls of radius $\frac{\rho}{4}$. Therefore, if $x_0, x_1, \ldots, x_p$ are $p+1$ points in $X$, then $d(x_i, x_j) \leq \frac{\rho}{4}$, for some $0 \leq i < j \leq p$.

Next, for $m, n \geq 1$, define $f_{m,n} : Y \to [0,1]$ by letting $f_{m,n}(y) = d(\theta_m(y), \theta_n(y))$. Then for all $\gamma \in \Gamma$ and $y \in Y$ we have that

$$|f_{m,n}(\gamma y) - f_{m,n}(y)| = |d(\theta_m(\gamma y), \theta_n(\gamma y)) - d(w(\gamma, y)\theta_m(y), w(\gamma, y)\theta_n(y))|$$

$$\leq d(\theta_m(\gamma y), w(\gamma, y)\theta_m(y)) + d(\theta_n(\gamma y), w(\gamma, y)\theta_n(y)).$$

From this it follows that $\int_Y |f_{m,n}(\gamma y) - f_{m,n}(y)| d\nu(y) \to 0$, for all $\gamma \in \Gamma$, as $m, n \to \infty$. Since $\|f_{m,n}\|_1 \leq 1$, for all $m, n$, the strong ergodicity assumption implies that $\|f_{m,n} - \int_Y f_{m,n}\|_1 \to 0$, as $m, n \to \infty$. In other words,

$$\int_Y |d(\theta_m(y), \theta_n(y)) - \tilde{d}(\theta_m, \theta_n)| d\nu(y) \to 0, \quad as \ m, n \to \infty.$$

Recalling that $\tilde{d}(\theta_n, \theta_m) \geq \varepsilon_0$, for all $l > k \geq 1$, it follows that

$$\nu(\{y \in Y|d(\theta_n(y), \theta_m(y)) > \frac{\varepsilon_0}{2}\}) \to 1, \quad as \ k, l \to \infty.$$

This further implies that

$$\nu(\{y \in Y|d(\theta_{n+i}(y), \theta_{n+j}(y)) > \frac{\varepsilon_0}{2}\}, for all 0 \leq i < j \leq p\}) \to 1, \quad as \ k \to \infty.$$

In particular, if $k$ is large enough, then we can find $y \in Y$ such that $d(\theta_{n+i}(y), \theta_{n+j}(y)) > \frac{\varepsilon_0}{2}$, for all $0 \leq i < j \leq p$. This contradicts the choice of $p$, and finishes the proof of the claim. \hfill \blacksquare

The following consequence of Lemma 2.4 might be of independent interest.

**Corollary 2.5.** Let $\Gamma \curvearrowright (Y, \nu)$ be a strongly ergodic p.m.p. action. Let $K$ a compact metrizable group endowed with a left-right invariant compatible metric $d$. Let $w : \Gamma \times Y \to K$ be a measurable cocycle, and $L < K$ be a closed subgroup. Assume that there exists a sequence of measurable maps $\phi_n : Y \to K$ such that $d(\phi_n(\gamma y)^{-1}w(\gamma, y)\phi_n(y), L) \to 0$, for all $\gamma \in \Gamma$ and almost every $y \in Y$.

Then there exists a measurable map $\phi : Y \to K$ such that $\phi(\gamma y)^{-1}w(\gamma, y)\phi(y) \in L$, for all $\gamma \in \Gamma$ and almost every $y \in Y$.

This result generalizes [Sc80] Proposition 2.3 which dealt with the case $L = \{e\}$ and $K = \mathbb{T}$. It also extends [Io13] Lemma J, where it was noticed that the proof of [Sc80] can be adapted to more generally treat the case when $L = \{e\}$ and $K$ is any compact metrizable group.

**Proof.** Endow $X = K/L$ with the metric $d(xL, yL) := d(x, yL) = \inf\{d(x, y\ell)|\ell \in L\}$. Then $(X, d)$ is a compact metric space and the left multiplication action $K \curvearrowright X$ is isometric. Define $\theta_n : Y \to X$ by letting $\theta_n(y) = \phi_n(y)K$. Then for all $\gamma \in \Gamma$ and almost every $y \in Y$ we have

$$\lim_{n \to \infty} \tilde{d}(\theta_n(\gamma y), w(\gamma, y)\theta_n(y)) = \lim_{n \to \infty} d(\phi_n(\gamma y)^{-1}w(\gamma, y)\phi_n(y), L) = 0.$$

By Lemma 2.4 we deduce that there exists a measurable map $\theta : Y \to X$ such that $\theta(\gamma y) = w(\gamma, y)\theta(y)$, for all $\gamma \in \Gamma$ and almost every $y \in Y$. If $\phi : Y \to K$ is a Borel map such that $\theta(y) = \phi(y)K$, for all $y \in Y$, then the conclusion follows. \hfill \blacksquare

**Lemma 2.6.** Let $\Gamma \curvearrowright (Y, \nu)$ be a strongly ergodic p.m.p. action and let $\Gamma \curvearrowright (X, \mu)$ be a distal p.m.p. action. Assume that $\alpha \prec \beta$, as witnessed by the asymptotically equivariant sequence $\theta_n : (Y, \nu) \to (X, \mu)$, $n \in \mathbb{N}$, of measure preserving maps. Then there exists a factor map $\theta : (Y, \nu) \to (X, \mu)$ from $\beta$ to $\alpha$, along with a subsequence $(\theta_{n_k})$ which converges weakly to $\theta$. 

Proof. Let \((\Gamma \curvearrowright^\alpha (X,\mu))\) be the distal tower associated to \(\alpha\), as in [1.1]-[1.3], and let \(\eta < \omega_1\) be the order of the tower. Since \(\alpha\) is distal we have \(\alpha = \alpha_\eta\). We prove the lemma by transfinite induction on \(\eta\). If \(\eta = 0\) then the statement is obvious, so assume that \(\eta > 0\).

Case 1: \(\eta = \eta_0 + 1\) is a successor ordinal. In this case the extension \((X,\mu) \to (X_{\eta_0},\mu_{\eta_0})\) is compact, so we may assume without loss of generality that \((X,\mu) = (X_{\eta_0},\mu_{\eta_0}) \otimes (K/L,m_{K/L})\), and that the action \(\alpha\) is of the form \(\gamma(x,kL) = (\gamma x,w(\gamma,x)kL)\) for some measurable cocycle \(w:\Gamma \times X_{\eta_0} \to K\). For each \(n\) we may write \(\theta_n(y) = (\theta_n^0(y),\theta_n^1(y))\), where \(\theta_n^0 : Y \to X_{\eta_0}\) and \(\theta_n^1 : Y \to K/L\) are the compositions of \(\theta_n\) with the left and right projections respectively. Applying the induction hypothesis to the action \(\alpha_{\eta_0}\) and the sequence \((\theta_n^0)\), we obtain a subsequence \((\theta_n^0)\) and a factor map \(\theta^0 : (Y,\nu) \to (X_{\eta_0},\mu_{\eta_0})\) such that \(\theta_n^0\) converges weakly to \(\theta^0\).

Define \(\bar{\theta}_n : (Y,\nu) \to (X,\mu)\) by \(\bar{\theta}_n(y) = (\theta_n^0(y),\theta_n^1(y))\), so that the subsequence \((\bar{\theta}_n)\) is asymptotically equivariant. Fix a compatible Polish metric \(d_0 \leq 1\) on \((X_{\eta_0},\mu_{\eta_0})\), let \(d_{K/L} \leq 1\) be a compatible \(K\)-invariant metric on \(K/L\), and let \(d\) be the compatible Polish metric on \(X\) given \(d((x,kL),(x',k'L)) = \frac{1}{2}(d_0(x,x') + d_{K/L}(kL,k'L))\). By Lemma [2.3] for each \(\gamma \in \Gamma\) we have

\[
\lim_{i \to \infty} \int_Y d(\bar{\theta}_n(\gamma y),\bar{\theta}_n(y)) \, d\nu(y) = 0.
\]

Applying Lemma [2.4], we obtain a measurable map \(\theta^0 : Y \to K/L\) with \(\theta^1(y) = w(\gamma,\theta^0(y))\theta^1(y)\), along with a subsequence \((\theta_n^0)\) of \((\bar{\theta}_n)\) with \(\int_Y d_{K/L}(\theta_n^0(y),\theta^0(y)) \, d\nu(y) \to 0\). Therefore, the map \(\theta : (Y,\nu) \to (X,\mu)\) defined by \(\theta(y) = (\theta^0(y),\theta^1(y))\) is a factor map from \(\beta\) to \(\alpha\), and by Lemma [2.3] the subsequence \((\bar{\theta}_n)\) converges weakly to \(\theta\).

Case 2: \(\eta\) is a limit ordinal. Fix a sequence \((\zeta_j)_{j \in \mathbb{N}}\) of ordinals which strictly increase to \(\eta\). For each \(j < k \in \mathbb{N}\) let \(\varphi_{j,k} : (X_{\zeta_j},\mu_{\zeta_j}) \to (X_{\zeta_k},\mu_{\zeta_k})\) denote the factor map from \(\alpha_{\zeta_j}\) to \(\alpha_{\zeta_k}\), and let \(\varphi_j : (X,\mu) \to (X_{\zeta_j},\mu_{\zeta_j})\) denote the factor map from \(\alpha = \alpha_{\eta}\) to \(\alpha_{\zeta_j}\). Applying the induction hypothesis to the action \(\alpha_{\eta}\) and the asymptotically equivariant sequence \(\varphi_0 \circ \theta_n : (Y,\nu) \to (X_{\zeta_0},\mu_{\zeta_0})\), we obtain a factor map \(\theta^0 : (Y,\nu) \to (X_{\zeta_0},\mu_{\zeta_0})\) and a subsequence \((n_i^0)_{i \in \mathbb{N}}\) such that \(\varphi_0 \circ \theta_{n_i^0}\) converges weakly to \(\theta^0\). Having defined the factor map \(\theta^0 : (Y,\nu) \to (X_{\zeta_0},\mu_{\zeta_0})\) and subsequence \((n_i^0)_{i \in \mathbb{N}}\), we apply the induction hypothesis to action \(\alpha_{\eta_{j+1}}\) and the sequence \((\varphi_{j+1} \circ \theta_{n_i^0})_{i \in \mathbb{N}}\) to obtain a factor map \(\theta^{j+1} : (Y,\nu) \to (X_{\zeta_{j+1}},\mu_{\zeta_{j+1}})\) and a subsequence \((n_i^{j+1})_{i \in \mathbb{N}}\) of \((n_i^0)_{i \in \mathbb{N}}\) with \(\varphi_{j+1} \circ \theta_{n_i^{j+1}}\) converging weakly to \(\theta^{j+1}\). Observe that \(\varphi_{j,k} \circ \theta^j = \theta^j\) for all \(j < k \in \mathbb{N}\). Since \(\alpha = \lim_j \alpha_{\zeta_j}\), this implies that there is a unique factor map \(\theta = \lim_j \theta^j\) from \(\alpha\) to \(\beta\) such that \(\varphi_j \circ \theta = \theta^j\) for all \(j \in \mathbb{N}\). Moreover, if we define the subsequence \((n_i)_{i \in \mathbb{N}}\) by \(n_i = n_i^0\), then for each \(j \in \mathbb{N}\), the sequence \((\varphi_j \circ \theta_{n_i})_{i \in \mathbb{N}}\) converges weakly to \(\theta_j\), and hence the sequence \((\theta_{n_i})_{i \in \mathbb{N}}\) converges weakly to \(\theta\).

Proof of Theorem [1.1]. This is immediate from Lemma [2.6].

Lemma 2.7. Let \(\Gamma \curvearrowright^\alpha (X,\mu)\) and \(\Gamma \curvearrowright^\beta (Y,\nu)\) be ergodic compact p.m.p. actions of \(\Gamma\). If \(\alpha\) are \(\beta\) are weakly isomorphic, then they are isomorphic.

Proof. Let \(\psi_0 : (X,\mu) \to (Y,\nu)\) and \(\psi_1 : (Y,\nu) \to (X,\mu)\) be factor maps from \(\alpha\) to \(\beta\) and from \(\beta\) to \(\alpha\) respectively. It suffices to show that the map \(\theta = \psi_1 \circ \psi_0\), factoring \(\alpha\) onto itself, is an isomorphism. Since \(\alpha\) is an ergodic compact action, we may assume that \((X,\mu) = (K/L,m_{K/L})\), where \(K\) is a compact metrizable group, \(L < K\) is a closed subgroup, and that \(\Gamma\) acts by translation on \(K/L\) via a homomorphism \(\Gamma \to K\) with dense image (in what follows we will identify \(\Gamma\) with its image in \(K\)). In order to show that \(\theta\) is injective on a conull subset of \(K/L\),
it is enough to show that the isometric linear embedding $T^\theta : L^2(K/L) \to L^2(K/L)$, $\xi \mapsto \xi \circ \theta$, which $\theta$ induces on $L^2(K/L)$, is surjective.

The translation action $K \actson K/L$ gives rise to a unitary representation $\lambda_{K/L}$ of $K$ on $\mathcal{H} = L^2(K/L)$, which we may view as a subrepresentation of the left regular representation of $K$ of $L^2(K)$ via the inclusion $L^2(K/L) \hookrightarrow L^2(K)$ associated to the natural projection $K \to K/L$. The representation $\lambda_{K/L}$ may be expressed as a direct sum $\lambda_{K/L} = \bigoplus_{\pi \in \hat{K}} \lambda^\pi_{K/L}$, where $\hat{K}$ is a collection of representatives for isomorphism classes of irreducible unitary representations of $K$, and where for each $\pi \in \hat{K}$, the representation $\lambda^\pi_{K/L}$ is the restriction of $\lambda_{K/L}$ to the closed linear span $\mathcal{H}^\pi$, of all subspaces of $\mathcal{H}$ on which $\lambda_{K/L}$ is isomorphic to $\pi$. For each $\pi \in \hat{K}$ we may further write $\mathcal{H}^\pi$ as a direct sum of irreducible subspaces $\mathcal{H}^\pi = \bigoplus_{i < n_\pi} \mathcal{H}^\pi,i$, where for each $i < n_\pi$ the representations $\lambda^\pi,i_{K/L} := \lambda^\pi_{K/L}|_{\mathcal{H}^\pi,i}$ is isomorphic to $\pi$. Since $\lambda_{K/L}$ is a subrepresentation of the left regular representation of $K$, by the Peter-Weyl Theorem, each $n_\pi$ is finite and therefore the subspaces $\mathcal{H}^\pi$ are all finite dimensional.

Since $\Gamma$ is dense in $K$, the operator $T^\theta$ intertwines $\lambda_{K/L}$ with itself, so by Schur’s Lemma we see that $T^\theta$ intertwines each $\lambda^\pi_{K/L}$ with itself, i.e., $T^\theta(\mathcal{H}^\pi) \subseteq \mathcal{H}^\pi$ for all $\pi \in \hat{K}$. Since $T^\theta$ is injective and each $\mathcal{H}^\pi$ is finite dimensional it follows that $T^\theta(\mathcal{H}^\pi) = \mathcal{H}^\pi$ for all $\pi \in \hat{K}$ and hence $T^\theta$ is surjective on $L^2(K/L)$, as was to be shown.

**Proof of Corollary 1.2** Assume that $\alpha$ is weakly equivalent to $\beta$. Note that this implies that $\alpha$ is strongly ergodic, since $\beta$ is strongly ergodic. Let $\alpha'$ and $\beta'$ be the maximal distal factors of $\alpha$ and $\beta$ respectively. As $\alpha'$ is a factor of $\alpha$, and $\alpha < \beta$, we have that $\alpha' < \beta$. Therefore, $\alpha'$ is a factor of $\beta$ by Theorem 1.1 and hence $\alpha'$ (being a distal factor of $\beta$) is a factor of $\beta'$. Likewise, reversing the roles of $\alpha$ and $\beta$, we obtain that $\beta'$ is a factor of $\alpha'$. Thus, $\alpha'$ and $\beta'$ are weakly isomorphic. The final statement of Corollary 1.2 now follows from Lemma 2.7.

**Proof of Corollary 1.3** Assume that $\alpha_{(a,b)} \prec \alpha_{(a',b')}$. Denote by $A, B$ and $A', B'$ the matrices constructed from the pairs $(a,b)$ and $(a',b')$. Since $A', B'$ have algebraic entries, $\alpha_{(a',b')}$ has spectral gap (see [BdS14, Theorem 1.2]). Corollary 1.2 gives that $\alpha_{(a,b)}$ is a factor of $\alpha_{(a',b')}$. Let $\theta : K \to K$ be a $\Gamma$-equivariant measurable map. If $t \in K$, then $K \ni x \mapsto \theta(x)^{-1}\theta(x)t \in K$ is a $\Gamma$-invariant map. Thus, there is $t = \delta(t) \in K$ such that $\theta(x)^{-1}\theta(x)t =\delta(t)$, for almost every $x \in K$. It follows that $\delta : K \to K$ is a continuous homomorphism whose restriction to $\Gamma$ is the identity.

In other words, $\delta$ satisfies $\delta(A) = A'$ and $\delta(B) = B'$. Since $K$ is a simple group, $\delta$ is one-to-one. Since $K$ is not isomorphic to any of its proper closed subgroups, $\delta$ is also onto. Thus, since $K$ has no outer automorphisms, we can find $g \in K$ such that $\delta(x) = gxg^{-1}$, for all $x \in K$. In particular, $A, A'$ must have the same trace, hence $\frac{a}{b} = \frac{a'}{b'}$. This completes the proof.

**Proof of Corollary 1.4** Assume by contradiction that $\Gamma \actson (K/L, m_{K/L})$ associated to a dense embedding of $\Gamma$ into a compact group $K$. Let $d$ be a compatible metric on $X = K/L$. By Lemma 2.3 we can find a sequence $\theta_n : Y \to X$ of measurable maps such that $d(\theta_n(\gamma y), \gamma \theta_n(y)) \to 0$, for all $\gamma \in \Gamma$ and almost every $y \in Y$. Denote by $\mathcal{R}_\sigma$ and $\mathcal{R}_\beta$ the equivalence relations associated to $\sigma$ and $\beta$, so that $\mathcal{R}_\sigma \subset \mathcal{R}_\beta$.

Since $\alpha$ is essentially free and $\alpha < \sigma$, $\sigma$ is essentially free. Since $\Gamma$ is non-amenable, the restriction of $\mathcal{R}_\sigma$ to any non-negligible set $Y_0 \subset Y$ is not hyperfinite. By [CI08, Theorem 1] we deduce the existence of a $\sigma(\Gamma)$-invariant non-negligible measurable set $Y_1 \subset Y$ such that the restriction $\sigma|_{Y_1}$ of $\sigma$ to $Y_1$ is strongly ergodic. By applying Lemma 2.4 to the restrictions of $\theta_n$ to $Y_1$ we conclude
that $\alpha$ is a factor of $\sigma_1$. Let $\theta : Y_1 \to X$ be a measurable, measure preserving map such that $\theta(\gamma y) = \gamma \theta(y)$, for every $\gamma \in \Gamma$ and almost every $y \in Y_1$.

We will reach a contradiction by applying [CPS11, Theorem 6.2]. To this end, we denote by $\Theta : L^\infty(X) \rtimes_\alpha \Gamma \to L^\infty(Y_1) \rtimes_{\sigma_1} \Gamma$ the $*$-homomorphism given by $\Theta(f) = f \circ \theta$ and $\Theta(u_\gamma) = u_\gamma$, for all $f \in L^\infty(X)$ and $\gamma \in \Gamma$. We view $N := L^\infty(Y) \rtimes_\beta \Lambda$ as a subalgebra of $M := L^\infty(Y) \rtimes_\beta \Lambda$. Let $p = 1_{Y_1} \in N$, $P = \Theta(L^\infty(X)) \subset pMp$ and $\mathcal{G} = \{aup | a \in \mathcal{U}(L^\infty(Y)), \gamma \in \Gamma\}$. Then $\mathcal{G}$ is a group of unitaries in $pMp$ which normalize $P$ and satisfies $\mathcal{G}'' = Np$.

Next, denote by $\tilde{M} = L(Z) \rtimes \Lambda$ the von Neumann algebra of the wreath product group $Z \rtimes \Lambda$. Recall that $(Y, \nu) = (Z, \rho)^\Lambda$ and consider a fixed embedding of $L^\infty(Z)$ into $L(Z)$. From this we get a $\Lambda$-equivariant embedding of $L^\infty(Y) = L^\infty(Z)^\Lambda$ into $L(\bigoplus_{\lambda \in \Lambda} Z) = L(Z)^\Lambda$, and thus an embedding $M \subset \tilde{M}$. It is easy to see that $(L^\infty(Y)p')' = p\tilde{M}p = L(\bigoplus_{\lambda \in \Lambda} Z)p$.

Thus, if we denote $Q = \mathcal{G}' \cap p\tilde{M}p$, then $Q \subset L(\bigoplus_{\lambda \in \Lambda} Z)p$. Since $Q$ commutes with $Np$ and $N$ has no amenable direct summand, [CI08, Theorem 2] implies that $Q$ is completely atomic.

Let $q \in Q$ be a non-zero projection such that $Qq = Cq$. Then $(\mathcal{G}q)' \cap q\tilde{M}q = Cq$. Moreover, since $\alpha$ is a compact action, the conjugation action $\mathcal{G} \rtimes P$ is compact. Hence, the conjugation action $\mathcal{G}q \rtimes Pq$ is compact and thus weakly compact (see [CPS11, Definition 6.1]). By applying [CPS11, Theorem 6.2] we get that either $(\mathcal{G}q)'' = Nq$ has an amenable direct summand or $P \subset \tilde{M} L(\Lambda)$ (see [CPS11, Definition 2.1]). Since $N$ has no amenable direct summand and $P \subset L(\bigoplus_{\lambda \in \Lambda} Z)$, neither of these conditions hold true, which gives the desired contradiction. 

\section*{References}


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