Math 220 Exam 1
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Name Solutions

There are 5 questions, for a total of 100 points. Point values are written beside each question.

1. Consider the statement: For all real numbers x and y, if x and y are irrational, then xy is irrational.

(a) [5 points] Write the converse of this statement.

For all real numbers x and y, if xy is irrational, then x and y are irrational.

(b) [5] Write the contrapositive of this statement.

For all real numbers x and y, if xy is rational, then x or y is rational.

(c) [5] Write the negation of this statement.

There exist real numbers x and y for which x and y are irrational and xy is rational.

(d) [5] Which of the above four statements *(the proposition, its converse (a), its contrapositive (b), its negation (c))* are true? (You need not justify your answer.)

(c) is true (Example $x = \sqrt{2}$, $y = \sqrt{2}$, so $xy = 2$.)
The original statement is false, and a counterexample is $x = \sqrt{2}$, $y = \sqrt{2}$.
The contrapositive is false since it is logically equivalent to the original statement (and the same counterexample shows this).
The converse is false, and a counterexample is $x = 2$, $y = \sqrt{2}$, so $xy = 2\sqrt{2}$.

2. [15] Prove that for all integers n, $n^2 + 3n$ is even.

Let n be an integer.

Case 1 n is even, that is, $n = 2k$ for some integer k.

Then $n^2 + 3n = (2k)^2 + 3(2k)$

$= 4k^2 + 6k$

$= 2(2k^2 + 3k)$, which is even.

Case 2 n is odd, that is, $n = 2l+1$ for some integer l.

Then $n^2 + 3n = (2l+1)^2 + 3(2l+1)$

$= 4l^2 + 4l + 1 + 6l + 3$

$= 4l^2 + 10l + 4$

$= 2(2l^2 + 5l + 2)$, which is even. $\square$

3. [30] Prove that for all integers m and n, mn is odd if and only if m is odd and n is odd.

This is a biconditional statement, so we must prove both:

(i) For all integers m and n, if mn is odd, then m is odd and n is odd.

(ii) For all integers m and n, if m is odd and n is odd, then mn is odd.

Proof of (i). We will prove the contrapositive, which is:

For all integers m and n, if m is even or n is even, then mn is even.

Proof: Let m and n be integers for which m is even or n is even.

Case 1. m is even, so that $m = 2k$ for some integer k. Then

$$mn = (2k)n = 2(kn), \text{ which is even.}$$

Case 2. n is even, so that $n = 2l$ for some integer l. Then

$$mn = m(2l) = 2(ml), \text{ which is even.}$$

Therefore, in either case, mn is even.

Proof of (ii). Let m and n be odd integers, so that $m = 2a+1$ and $n = 2b+1$ for some integers a and b. Then

$$mn = (2a+1)(2b+1) = 4ab + 2a + 2b + 1$$
$$= 2(2ab + a + b) + 1, \text{ which is odd.} \quad \square$$

4. [15] Prove that there do not exist integers m and n for which $6m - 14n = 7$.

Proof by contradiction:

Assume there do exist integers m and n for which $6m - 14n = 7$.

Then

$$2(3m - 7n) = 7.$$

The left side of the equation is an even integer, while the right side is an odd integer. This is a contradiction. Therefore there do not exist integers m and n for which $6m - 14n = 7$. $\square$

5. [20] Prove by induction that for each positive integer n,

$$P(n): \quad 1+3+5+\cdots+(2n-1)=n^2.$$

1. Check that $P(1)$ is true: $2(1)-1=1$ and $1^2=1$, so $P(1)$ is true.
2. Assume that for some positive integer m, $P(m)$ is true, that is,

$$1+3+5+\cdots+(2m-1)=m^2. \quad (*)$$

Then by the induction hypothesis $(*)$,

$$\begin{aligned} &1+3+5+\cdots+(2(m+1)-1) \\ &= 1+3+5+\cdots+(2m-1)+(2(m+1)-1) \\ &= m^2+(2(m+1)-1) \\ &= m^2+2m+2-1 \\ &= m^2+2m+1 \\ &= (m+1)^2. \end{aligned}$$

Therefore $P(m+1)$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n. $\square$