Math 220 Practice for Exam 1

1. Consider the statement: For all integers m and n, if m is even and n is even , then $m + n$ is divisible by 4. False. For example, take $m = 2$ and $n = 4$. Then $m+n = 6$.

(a) Write the converse of this statement.

For all integers m and n, if m+n is divisible by 4, then m is even and n is even.

False. Counterexample: $m = 1$, $n = 3$, $m+n = 4$.

(b) Write the contrapositive of this statement.

For all integers m and n, if m+n is not divisible by 4, then m is odd or n is odd.

False. (The contrapositive is equivalent to the original statement, so it is false since the original statement is false. The same counterexample shows that it is false: $m = 2$, $n = 4$.)

(c) Write the negation of this statement.

There exist integers m and n such that m is even and n is even and m+n is not divisible by 4.

True. (For example, $m = 2$, $n = 4$.)

(d) [5] Which of the above four statements *(the proposition, its converse (a), its contrapositive (b), its negation (c))* are true? (You need not justify your answer.)

$\neg(P \Rightarrow Q)$ is logically equivalent to $P \wedge (\neg Q)$

2. Consider the statement: For all real numbers x and y, if $\underbrace{xy \text{ is rational}}_{P}$, then $\underbrace{x \text{ is rational}}_{Q}$.

False
$x = \sqrt{2}$
$y = \sqrt{2}$
$xy = 2$

(a) Write the converse of this statement.

For all real numbers x and y,
if x is rational, then xy is rational.

False
$y = \sqrt{2}$
$x = 2$
$xy = 2\sqrt{2}$

(b) Write the contrapositive of this statement.

for all real numbers x and y,
if x is irrational, then xy is irrational.

False
(logically equivalent statement, to the original since the original statement is false, so it is false.)

(c) Write the negation of this statement.

There exist real numbers x and y such that
xy is rational and x is irrational.

True
$x = \sqrt{2}$
$y = \sqrt{2}$
$xy = 2$

(d) Which of the above four statements *(the proposition, its converse (a), its contrapositive (b), its negation (c))* are true? (You need not justify your answer.)

3. Prove that for all integers m and n, if m and n are both odd, then $m+n$ is even. Is the converse true?

Proof: Let m and n be odd integers. Then $m = 2k+1$ and $n = 2l+1$ for some integers k and l. It follows that

$$\begin{aligned} m+n &= (2k+1)+(2l+1) \\ &= 2k+2l+2 \\ &= 2(k+l+1), \end{aligned}$$

which is even. $\square$

The converse states that for all integers m and n, if $m+n$ is even, then m and n are both odd.
This is false. A counterexample is $m=2$, $n=2$.

biconditional

4. Prove that for all integers n, n is divisible by 3 if, and only if, n^2 is divisible by 3.

⊗ (Use from book p. 43: n is not divisible by 3 if, and only if, $n = 3k+1$ or $n = 3k+2$ for some integer k.)

($\Rightarrow$) For all integers n, if n is divisible by 3, then n^2 is divisible by 3.

Proof: Let n be an integer that is divisible by 3, so $n = 3k$ for some integer k. It follows that $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$, which is divisible by 3.

($\Leftarrow$) For all integers n, if n^2 is divisible by 3, then n is divisible by 3.

Proof: We'll prove the contrapositive, i.e. we'll prove that for all integers n, if n is ~~~~ NOT divisible by 3, then n^2 is NOT divisible by 3.

Let n be an integer that is not divisible by 3, so $n = 3k+1$ or $n = 3k+2$ for some integer k. (by ⊗)

∴

Case 1 $n = 3k+1$. Then $n^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$, so n^2 is not divisible by 3 (by ⊗).

Case 2 $n = 3k+2$. Then $n^2 = (3k+2)^2 = 9k^2+12k+4$
$= 3(3k^2+4k+1)+1$,

so n^2 is not divisible by 3. (by ⊗).

5. Prove there do not exist integers m and n for which $9m + 51n = 2$.

Proof by contradiction:

Assume that m and n are integers for which $9m+51n=2$. Note that $9m+51n = 3(3m+17n)$, so $9m+51n$ is divisible by 3. However, 2 is not divisible by 3. This is a contradiction. Therefore, there do not exist integers m and n for which $9m+51n=2$. $\square$

6. Prove by induction that for each positive integer n,

$$P(n): \quad 2+6+10+\cdots+(4n+2)=2(n+1)^2.$$

Proof:

1. We check that $P(1)$ is true: When $n=1$, the left side of the equation is $2+6=8$ (since $4\cdot 1+2=6$). The right side of the equation is $2(1+1)^2 = 2\cdot 2^2 = 2\cdot 4 = 8$. So $P(1)$ is true.

2. Assume that for some positive integer m, $P(m)$ is true, that is, $2+6+10+\cdots+(4m+2) = 2(m+1)^2$. We wish to prove that $P(m+1)$ is true, that is, we wish to prove that $2+6+10+\cdots+(4(m+1)+2) = 2((m+1)+1)^2$.

First we rewrite the left side, including the term corresponding to m:

$$2+6+10+\cdots+(4(m+1)+2)$$

$$= \underbrace{2+6+10+\cdots+(4m+2)}+(4(m+1)+2)$$

By the inductive hypothesis,

$$2+6+10+\cdots+(4m+2) = 2(m+1)^2.$$

So this expression is equal to

$$\begin{aligned} &2(m+1)^2 + (4(m+1)+2) \\ &= 2(m^2+2m+1)+(4m+4+2) \\ &= 2m^2+4m+2+4m+6 \\ &= 2m^2+8m+8 \\ &= 2(m^2+4m+4) \\ &= 2(m+2)^2 \\ &= 2((m+1)+1)^2. \end{aligned}$$

Therefore $P(m+1)$ is true.

It follows, by the Principle of Mathematical Induction, that $P(n)$ is true for all positive integers n. $\square$

7. Prove by induction that for all integers $n \geq 1$, 3 divides $n^3 + 2n$.

Proof Let $P(n)$ be the statement that 3 divides n^3+2n.

1. $P(1)$: $1^3+2\cdot 1 = 1+2 = 3$, which is divisible by 3. So $P(1)$ is true.

2. Assume that for some positive integer m, 3 divides m^3+2m, that is, $m^3+2m = 3k$ for some integer k.

We want to use this to prove that $P(m+1)$ is true, that is that 3 divides $(m+1)^3 + 2(m+1)$.

Note that

$$
\begin{aligned}
(m+1)^3 + 2(m+1) &= m^3 + 3m^2 + 3m + 1 + 2m + 2 \\
&= (m^3 + 2m) + (3m^2 + 3m + 3) \\
&= 3k + 3(m^2+m+1) \\
&= 3(k+m^2+m+1),
\end{aligned}
$$

which is divisible by 3. So $P(m+1)$ is true.

It follows that $P(n)$ is true for all positive integers n. $\square$