Math 220 Exam 2
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S. Witherspoon

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There are 7 questions, for a total of 100 points. Point values are written beside each question.

1. [15 points] Let $a_1 = 1$, $a_2 = 5$, and $a_{n+1} = 5a_n - 6a_{n-1}$ for all $n \geq 2$. Prove that for all positive integers n, $a_n = 3^n - 2^n$.

2. Consider the following two sets:

$$\begin{aligned} A &= \{n \in \mathbb{Z} \mid n = 3i - 1 \text{ for some } i \in \mathbb{Z}\} \\ B &= \{n \in \mathbb{Z} \mid n = 6j + 2 \text{ for some } j \in \mathbb{Z}\} \end{aligned}$$

(a)[6] List at least 5 elements of A and at least 5 elements of B.

(b) [7] Is $A \subseteq B$? Prove or disprove.

(c) [7] Is $B \subseteq A$? Prove or disprove.

3. [20] Let A and B be subsets of a universal set U. Prove that $A - B = A \cap \overline{B}$.

4. For each $i \in \mathbb{Z}^+$, let $A_i = \left[-\frac{1}{i},\ i^2\right]$.

(a) [5] Find $A_1 \cap A_2$ and $A_1 \cup A_2$.

(b) [10] Find $\bigcap_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty} A_i$.

5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ n+1, & \text{if } n \text{ is odd} \end{cases}$

(a) [5] Is f one-to-one? Justify your answer.

(b) [5] Is f onto? Justify your answer.

6. [10] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (-y, x) \quad \text{and} \quad g(x, y) = (x + 2, y - 1)$$

for all $(x, y) \in \mathbb{R}^2$. Find $f \circ g$ and $g \circ f$. (That is, find formulas for $(f \circ g)(x, y)$ and $(g \circ f)(x, y)$.)

7. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

Is f invertible? If so, find f^{-1}. If not, explain why not.