Math 220 Exam 2
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Name Solutions

There are 7 questions, for a total of 100 points. Point values are written beside each question.

1. [15 points] Let $a_1 = 1$, $a_2 = 5$, and $a_{n+1} = 5a_n - 6a_{n-1}$ for all $n \geq 2$. Prove that for all positive integers n, $a_n = 3^n - 2^n$.

Proof by strong induction:

1. $3^1 - 2^1 = 3 - 2 = 1 = a_1$ and

$3^2 - 2^2 = 9 - 4 = 5 = a_2$,

so the formula holds for $n=1$ and $n=2$.

2. Assume that there is a positive integer m such that

$$a_k = 3^k - 2^k$$

for all k for which $1 \leq k \leq m$.

Then

$$\begin{aligned} a_{m+1} &= 5a_m - 6a_{m-1} \\ &= 5(3^m - 2^m) - 6(3^{m-1} - 2^{m-1}) \\ &= 5\cdot 3^m - 5\cdot 2^m - 2\cdot 3\cdot 3^{m-1} + 3\cdot 2\cdot 2^{m-1} \\ &= \underline{5\cdot 3^m} - \underline{\underline{5\cdot 2^m}} - \underline{2\cdot 3^m} + \underline{\underline{3\cdot 2^m}} \\ &= (5-2)3^m + (3-5)2^m \\ &= 3\cdot 3^m - 2\cdot 2^m \\ &= 3^{m+1} - 2^{m+1}. \end{aligned}$$

By strong math induction, $a_n = 3^n - 2^n$ for all positive integers n.

2. Consider the following two sets:

$$A = \{n \in \mathbb{Z} \mid n = 3i - 1 \text{ for some } i \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 6j + 2 \text{ for some } j \in \mathbb{Z}\}$$

(a)[6] List at least 5 elements of A and at least 5 elements of B.

Some elements of A: $-1, 2, 5, 8, 11$

Some elements of B: $2, 8, 14, 20, 26$

(b) [7] Is $A \subseteq B$? Prove or disprove.

No: $5 \in A$ and $5 \notin B$

Proof that $5 \notin B$: If it were, then

$5 = 6j + 2$ for some integer j,

so $6j = 5 - 2 = 3$,

and so $j = \frac{3}{6} = \frac{1}{2}$.

However, $\frac{1}{2}$ is not an integer, so this is a contradiction.

Therefore $5 \notin B$.

(c) [7] Is $B \subseteq A$? Prove or disprove.

Yes: Let $n \in B$, so that $n = 6j + 2$ for some $j \in \mathbb{Z}$.

Then $n = 3(2j) + 3 - 3 + 2$

$= 3(2j + 1) - 1$.

Letting $i = 2j + 1$, we see that $n = 3i - 1$, and so $n \in A$.

Therefore $B \subseteq A$.

3. [20] Let A and B be subsets of a universal set U. Prove that $A - B = A \cap \overline{B}$.

Proof that $A-B \subseteq A \cap \overline{B}$:

Let $a \in A - B$. Then $a \in A$ and $a \notin B$.
So $a \in A$ and $a \in \overline{B}$, in other words, $a \in A \cap \overline{B}$.

Proof that $A \cap \overline{B} \subseteq A-B$:

Let $a \in A \cap \overline{B}$. Then $a \in A$ and $a \in \overline{B}$.
So $a \in A$ and $a \notin B$, in other words, $a \in A-B$.

Since we have shown that $A-B \subseteq A \cap \overline{B}$ and $A \cap \overline{B} \subseteq A-B$,
we conclude that $A-B = A \cap \overline{B}$.

4. For each $i \in \mathbb{Z}^+$, let $A_i = \left[-\frac{1}{i},\ i^2\right]$.

(a) [5] Find $A_1 \cap A_2$ and $A_1 \cup A_2$.

$A_1 = [-1, 1]$, $A_2 = [-\frac{1}{2}, 4]$, so

$A_1 \cap A_2 = [-\frac{1}{2}, 1]$ and $A_1 \cup A_2 = [-1, 4]$

(b) [10] Find $\bigcap_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty} A_i$.

$$\bigcap_{i=1}^{\infty} A_i = [0, 1]$$

$$\bigcap_{i=1}^{\infty} A_i = [-1, \infty)$$

5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ n+1, & \text{if } n \text{ is odd} \end{cases}$

(a) [5] Is f one-to-one? Justify your answer.

No: $f(2) = 4$ and $f(3) = 4$

so $f(2) = f(3)$ and $2 \neq 3$

(b) [5] Is f onto? Justify your answer.

No: $3 \notin \text{ran} f$

(In fact, $\text{ran} f = 2\mathbb{Z}$ (all even integers) since for all even integers m, we may write $m = 2k$ for some integer k. Let $n = 2k-1$, an odd integer. Then $f(n) = f(2k-1) = 2k-1+1 = 2k = m$ since $2k-1$ is odd. Therefore $m \in \text{ran} f$. By definition of f, $\text{ran} f \subseteq 2\mathbb{Z}$. So $\text{ran} f = 2\mathbb{Z}$.)

6. [10] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (-y, x) \quad \text{and} \quad g(x, y) = (x + 2, y - 1)$$

for all $(x, y) \in \mathbb{R}^2$. Find $f \circ g$ and $g \circ f$. (That is, find formulas for $(f \circ g)(x, y)$ and $(g \circ f)(x, y)$.)

$(f \circ g)(x,y) = f(g(x,y)) = f(x+2, y-1) = (-y+1, x+2)$

$(g \circ f)(x,y) = g(f(x,y)) = g(-y, x) = (-y+2, x-1)$

7. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

Is f invertible? If so, find f^{-1}. If not, explain why not.

Yes: $f^{-1}(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0 \end{cases}$

To see this, find the formula separately for $x \geq 0$ and for $x < 0$:

$\underline{x \geq 0}$ Write $y = \sqrt{x}$ and interchange x and y to get

$x = \sqrt{y}$, then solve for y:

$y = x^2$

$\underline{x < 0}$ write $y = -x^2$ and interchange x and y to get

$x = -y^2$, then solve for y:

$-x = y^2$

$y = \pm\sqrt{-x}$

Now determine which of $\sqrt{-x}$ or $-\sqrt{-x}$ is the correct value. In order for $(f^{-1} \circ f)(x)$ to equal x when $x < 0$, we take $-\sqrt{-x}$.