Math 220 Practice for Exam 2

1. Let $a_1 = 1$, $a_2 = 9$, and $a_{n+1} = 9a_n - 20a_{n-1}$ for all $n \geq 2$. Prove that for all positive integers n, $a_n = 5^n - 4^n$.

Proof (by strong induction):

1. Check the formula is valid when $n=1, n=2$:

   $a_1 = 1 = 5^1 - 4^1$ ✓ $\quad a_2 = 9 = 5^2 - 4^2$ ✓

2. Assume that for some integer m, $m \geq 2$, $a_k = 5^k - 4^k$ for all k such that $1 \leq k \leq m$.

Then

$$a_{m+1} = 9a_m - 20a_{m-1} \leftarrow \text{given information}$$

$$= 9(5^m - 4^m) - 20(5^{m-1} - 4^{m-1}) \leftarrow \text{by induction assumption}$$

$$= 9(5^m - 4^m) - \underbrace{20}_{20 = 4\cdot 5}\cdot 5^{m-1} + \underbrace{20}_{20 = 5\cdot 4}\cdot 4^{m-1}$$

WANT this to $= 5^{m+1} - 4^{m+1}$

$$= 9(5^m - 4^m) - 4\cdot 5\cdot 5^{m-1} + 5\cdot 4\cdot 4^{m-1}$$

$$= 9(5^m - 4^m) - 4\cdot 5^m + 5\cdot 4^m$$

$$= 9\cdot 5^m - 9\cdot 4^m - 4\cdot 5^m + 5\cdot 4^m$$

$$= (9-4)\cdot 5^m + (-9+5)\cdot 4^m$$

$$= 5\cdot 5^m + (-4)\cdot 4^m$$

$$= 5^{m+1} - 4^{m+1}.$$

Therefore $a_{m+1} = 5^{m+1} - 4^{m+1}$. So $a_n = 5^n - 4^n$ for all positive integers n. $\square$

2. Let $a_1 = 3$, $a_2 = 5$, and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for all $n \geq 2$. Prove that for all positive integers n, $3 \leq a_n \leq 5$.

Proof (by strong induction):

1. Check that the statement is true where $n=1$, $n=2$:

$a_1 = 3$ so $3 \leq a_1 \leq 5$ ✓

$a_2 = 5$ so $3 \leq a_2 \leq 5$ ✓

2. Assume that for some integer m, $m \geq 2$,

$3 \leq a_k \leq 5$ for all k such that $1 \leq k \leq m$.

Then $3 \leq a_m \leq 5$ and $3 \leq a_{m-1} \leq 5$, so

$$3+3 \leq a_m + a_{m-1} \leq 5+5$$

$$6 \leq a_m + a_{m-1} \leq 10$$

$$3 \leq \frac{1}{2}(a_m + a_{m-1}) \leq 5$$

$$3 \leq a_{m+1} \leq 5.$$

By strong induction, $3 \leq a_n \leq 5$ for all positive integers n.

3. Consider the following two sets:

$$S = \{n \in \mathbb{Z} \mid n = 3x + 6y \text{ for some } x, y \in \mathbb{Z}\},$$
$$T = \{n \in \mathbb{Z} \mid n = 3x + 2y \text{ for some } x, y \in \mathbb{Z}\} = \{n \in \mathbb{Z} \mid n = 3a + 2b \text{ for some } a, b \in \mathbb{Z}\}$$

(a) Is $S \subseteq T$? Justify your answer. yes: let $n \in S$ so that $n = 3x + 6y$ for some $x, y \in \mathbb{Z}$. Note that

$$n = 3x + 6y = 3x + 2(3y),$$

so setting $a = x$ and $b = 3y$, we have $n = 3a + 2b$ for $a, b \in \mathbb{Z}$. It follows that $n \in T$. Therefore $S \subseteq T$.

(b) Is $T \subseteq S$? Justify your answer. No: For example, $5 \in T$ since $5 = 3 + 2$, and $5 \notin S$. (Note that elements of S are all multiples of 3, since $3x + 6y = 3(x + 2y)$, and 5 is not a multiple of 3.)

4. Consider the following statement.

P: For all sets A and B, $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

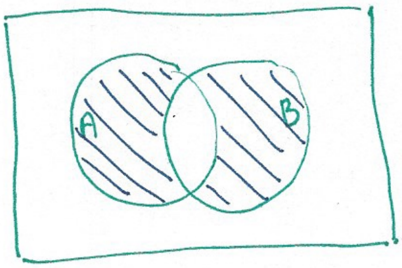
(a) *Just for this part*, let $A = \{1, 2, 3, 4\}$ and $B = \{0, 2, 4\}$. Find the following sets:

$A \cup B = \{0, 1, 2, 3, 4\}$ $\quad$ $A \cap B = \{2, 4\}$

$A - B = \{1, 3\}$ $\quad$ $B - A = \{0\}$

$(A \cup B) - (A \cap B) = \{0, 1, 3\}$ $\quad$ $(A - B) \cup (B - A) = \{0, 1, 3\}$

(b) Draw a Venn diagram to illustrate the statement P in general.



(c) Prove the statement P.

Proof that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$:

Let $x \in (A \cup B) - (A \cap B)$. So $x \in A$ or $x \in B$ and $x \notin A \cap B$.

Case 1 $x \in A$. Since $x \notin A \cap B$, it follows that $x \notin B$. Therefore $x \in A - B$.

Case 2 $x \in B$. Since $x \notin A \cap B$, it follows that $x \notin A$. Therefore $x \in B - A$.

In either case, $x \in (A - B) \cup (B - A)$.

Therefore $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$.

Proof that $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$:

Let $x \in (A - B) \cup (B - A)$, that is, $x \in A - B$ or $x \in B - A$.

Case 1 $x \in A - B$. Then $x \in A$ and $x \notin B$. Since $x \notin B$, it follows that $x \notin A \cap B$. Therefore $x \in A \cup B$ and $x \notin A \cap B$, that is, $x \in (A \cup B) - (A \cap B)$.

Case 2 $x \in B - A$. Then $x \in B$ and $x \notin A$. Since $x \notin A$, it follows that $x \notin A \cap B$. Therefore $x \in A \cup B$ and $x \notin A \cap B$, that is, $x \in (A \cup B) - (A \cap B)$.

In either case, $x \in (A \cup B) - (A \cap B)$.

Therefore $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

Since both subset containments hold, we conclude that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$. $\square$

5. (a) [5] Let $B = \{2, 3, 5, 8\}$ and $C = \{3, 7\}$. Find $B \times C$ (that is, write out all the elements of this set).

$\{ (2,3), (3,3), (5,3), (8,3), (2,7), (3,7), (5,7), (8,7) \}$

(b) Prove that *for all sets A, B, and C*, if $A \subseteq B$, then $A \times C \subseteq B \times C$.

Assume that A, B, C are sets for which $A \subseteq B$. Let $(a,c) \in A \times C$, that is, $a \in A$ and $c \in C$. Since $A \subseteq B$, it follows that $a \in B$. Therefore $(a,c) \in B \times C$. We have shown that $A \times C \subseteq B \times C$. $\square$

6. For each positive integer i, let $A_i = \left[-\frac{1}{i}, \frac{i}{i+1}\right]$. In the following, you need not prove that your answers are correct.
(a) Find $A_1 \cup A_2$ and $A_1 \cap A_2$.

$A_1 \cup A_2 = [-1, \frac{1}{2}] \cup [-\frac{1}{2}, \frac{2}{3}] = [-1, \frac{2}{3}]$

$A_1 \cap A_2 = [-1, \frac{1}{2}] \cap [-\frac{1}{2}, \frac{2}{3}] = [-\frac{1}{2}, \frac{1}{2}]$

(b) Find $\bigcup_{i \in \mathbb{Z}+} A_i$ and $\bigcap_{i \in \mathbb{Z}+} A_i$.

$\bigcup_{i \in \mathbb{Z}^+} A_i = [-1, 1)$

$\bigcap_{i \in \mathbb{Z}^+} A_i = [0, \frac{1}{2}]$

7. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = \begin{cases} n-2, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd} \end{cases}$

(a) Is f injective? Justify your answer.

No: $f(4) = f(1)$

(b) What is the range of f? Justify your answer.

ran $f = 2\mathbb{Z}$ (that is, all even integers)

we will first show that $2\mathbb{Z} \subseteq$ ran f:

To see this, let $m \in 2\mathbb{Z}$, that is, $m = 2k$ for some integer k.
(We want to show that $m = f(n)$ for some integer n.)
Let $n = 2k+2$. Then n is even, so $f(n) = n-2 = 2k+2-2 = 2k$,
that is, $f(n) = m$, as desired. Therefore $2\mathbb{Z} \subseteq$ ran f.

Now we will show that ran $f \subseteq 2\mathbb{Z}$: Let $m \in$ ran f, i.e. $m = f(n)$ for some integer n.

Case 1 n is even. Then $m = f(n) = n-2$, which is even.

Case 2 n is odd. Then $m = f(n) = 2n$, which is even.

In either case, we see that m is even. Therefore $m \in 2\mathbb{Z}$.
It follows that ran $f \subseteq 2\mathbb{Z}$.

We have shown that $2\mathbb{Z} \subseteq$ ran f, and ran $f \subseteq 2\mathbb{Z}$.

Therefore ran $f = 2\mathbb{Z}$. $\square$

8. Let $A = \{r, s, t\}$. Let f and g be the functions from A to A defined by

$$f(r) = t, \quad f(s) = r, \quad f(t) = s,$$
$$g(r) = s, \quad g(s) = t, \quad g(t) = t.$$

(a) Find the function $f \circ g$ (i.e., specify its values on each element of A).

$(f \circ g)(r) = f(g(r)) = f(s) = r$

$(f \circ g)(s) = f(g(s)) = f(t) = s$

$(f \circ g)(t) = f(g(t)) = f(t) = s$

(b) Which of the two functions f, g is invertible? For each that is invertible, find its inverse function (by specifying its values on each element of A).

f is invertible: $f^{-1}(r) = s$, $f^{-1}(s) = t$, $f^{-1}(t) = r$

g is not invertible:

- g is not one-to-one since $g(s) = g(t)$.
- It also is not onto since $r \notin \operatorname{ran} g$.

9. For each of the following functions, state whether it is invertible or not. You need not justify your answer.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x + 2$ for all $x \in \mathbb{Z}$.

No.

f is not ~~1-1~~: onto:

$3 \notin \operatorname{ran} f$ since if $3 = 3x + 2$

$1 = 3x$

$\frac{1}{3} = x$ $\quad \frac{1}{3} \notin \mathbb{Z}$

so $3 \notin \operatorname{ran} f$

(b) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 3x + 2$ for all $x \in \mathbb{Q}$.

yes, f is invertible: $y = 3x+2$

interchange x and y, then solve for y: $x = 3y+2$

$3y = x - 2$

$y = \frac{x-2}{3}$

$f^{-1}(x) = \frac{x-2}{3}$

(c) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = x^3$ for all $x \in \mathbb{Q}$.

No. f is not onto:

$2 \notin \operatorname{ran} f$ since if $2 = x^3$ then $x = \sqrt[3]{2}$, which is not rational.

(d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ for all $x \in \mathbb{R}$.

yes, f is invertible: set $y = x^3$,

interchange x, y: $x = y^3$

solve for y: $y = \sqrt[3]{x}$

$f^{-1}(x) = \sqrt[3]{x}$