Math 220 Final Exam Practice Problems
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The following are just a few representative problems. They are not meant to include examples of all possible problems that may be on the exam. You should also be prepared to work any problems similar to homework, examples from class, and the first two exams.

rational numbers $\mathbb{Q} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \}$

1. Prove or find a counterexample: irrational means not rational

(a) For all real numbers x, x is irrational if, and only if, $10x$ is irrational.

Proof: ($\Rightarrow$) We'll prove that for all real numbers x, if x is irrational, then $10x$ is irrational.

We'll prove this by contrapositive, i.e. we'll prove that for all real numbers x, if $10x$ is rational, then x is rational.

Let x be a real number for which $10x$ is rational, i.e. $10x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$. So $x = \frac{a}{10b}$, and it follows that x is rational.

($\Leftarrow$) Contrapositive: We'll prove that for all real numbers x, if x is rational, then $10x$ is rational.

Let x be a real number that is rational, i.e. $x = \frac{a}{b}$ for some integers a, b, with $b \neq 0$. Then $10x = \frac{10a}{b}$, which is rational. $\square$

(b) For all real numbers x, x is irrational if, and only if, $\sqrt{2}x$ is irrational.

False.

Counterexample to ($\Rightarrow$): $x = \sqrt{2}$ ($\sqrt{2}$ is irrational and $\sqrt{2} \cdot \sqrt{2} = 2$ is rational)

Remember: The negation of an implication is not an implication!

$\neg(P \Rightarrow Q)$ is $P \wedge (\neg Q)$

2. Prove that $\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}$ for all natural numbers n.

Proof by math induction:

1. Check $n=1$: $\sum_{i=1}^{1} i(i+2) = 1(1+2) = 3$ ✓

$$\frac{1(1+1)(2\cdot 1+7)}{6} = \frac{18}{6} = 3 \checkmark \qquad 3=3 \checkmark$$

2. Assume that for some positive integer m, $\sum_{i=1}^{m} i(i+2) = \frac{m(m+1)(2m+7)}{6}$.

Then

$$\sum_{i=1}^{m+1} i(i+2) = \sum_{i=1}^{m} i(i+2) + (m+1)((m+1)+2)$$

$$= \frac{m(m+1)(2m+7)}{6} + (m+1)((m+1)+2) \quad \text{(by the induction hypothesis)}$$

$$= (m+1)\left(\frac{m(2m+7)}{6} + \frac{m+3}{} \cdot \frac{6}{6}\right)$$

$$= (m+1)\left(\frac{m(2m+7)+6(m+3)}{6}\right)$$

$$= (m+1)\left(\frac{2m^2+7m+6m+\cancel{3}\,18}{6}\right)$$

$$= (m+1)\left(\frac{2m^2+13m+\cancel{3}\,18}{6}\right)$$

$$= (m+1)\frac{(2m+9)(m+2)}{6}$$

$$= \frac{(m+1)(m+2)(2(m+1)+7)}{6}$$

Therefore, by math induction, $\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}$ for all positive integers n.

3. Let A, B, and C be sets contained in a universal set. Prove that

$$(A - B) - C = A - (B \cup C).$$

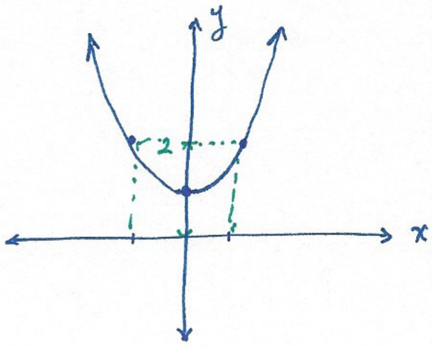
Proof that $(A-B)-C \subseteq A-(B\cup C)$:

Let $x \in (A-B)-C$, that is, $x \in A-B$ and $x \notin C$.
It follows that $x \in A$ and $x \notin B$ and $x \notin C$. Since $x \notin B$ and $x \notin C$, we have $x \notin B\cup C$. It follows that $x \in A - (B \cup C)$.

Proof that $A-(B\cup C) \subseteq (A-B)-C$:

Let $x \in A-(B\cup C)$, that is, $x \in A$ and $x \notin B\cup C$.
Since $x \notin B \cup C$, we have $x \notin B$ and $x \notin C$. So $x \in A$ and $x \notin B$ and $x \notin C$, and therefore $x \in (A-B)-C$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$.
(a) Find $f^{-1}((0,2))$ (where $(0,2)$ denotes an open interval).



$f^{-1}((0,2)) = \{ x \in \mathbb{R} \mid f(x) \in (0,2) \}$

i.e. $f^{-1}((0,2)) = \{ x \in \mathbb{R} \mid 0 < x^2+1 < 2 \}$

So $f^{-1}((0,2)) = (-1,1)$

(b) Find the range of f.

$\operatorname{ran} f = [1, \infty)$

(c) Is f one-to-one? Justify your answer.

No : $f(1) = f(-1)$

5. Let A and B be sets, and let $f : A \to B$ be a function. Let X be a subset of A, and let Y be a subset of B for which $f(X) \subseteq Y$.
(a) Prove that $X \subseteq f^{-1}(Y)$.

By definition, $f^{-1}(Y) = \{ x \in X \mid f(x) \in Y \}$ and $f(X) = \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}$

Proof: Let $x \in X$. Then $f(x) \in f(X)$. Since $f(X) \subseteq Y$, it follows that $f(x) \in Y$. So $x \in f^{-1}(Y)$.
Therefore $X \subseteq f^{-1}(Y)$. $\square$

(b) If $f(X) = Y$, is it necessarily true that $X = f^{-1}(Y)$? Justify your answer.

No.

Counterexample: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$X = \{1\}$ $\quad$ $Y = \{1\}$

$f(X) = Y$ $\quad$ $f^{-1}(Y) = \{-1, 1\}$

6. Find a solution to the equation $14x + 18y = 114$ in which x and y are integers.

gcd: $(14, 18) = 2$

Euclidean Algorithm:

$18 = 14 \cdot 1 + 4$

$14 = 4 \cdot 3 + 2 \leftarrow 2 = 14 - 4 \cdot 3$

$4 = 2 \cdot 2$

$2 = 14 - (18 - 14) \cdot 3$

$2 = 14 - 18 \cdot 3 + 14 \cdot 3$

$2 = 14 \cdot 4 + 18(-3)$

multiply by 57:

$114 = 14 \cdot 4 \cdot 57 + 18(-3)(57)$

$= 14(228) + 18(-171)$

$x = 228, \quad y = -171$

7. Let $A = \{1, 2, 3\}$ and let X be the set of all bijective functions $f : A \to A$. Define a relation R on X by fRg if and only if $f(1) = g(1)$.

(a) Prove that R is an equivalence relation.

R is reflexive: Let $f \in X$. Since $f(1) = f(1)$, it follows that $f R f$.

R is symmetric: Let $f, g \in X$ be such that fRg, that is, $f(1) = g(1)$. Then $g(1) = f(1)$, so $g R f$.

R is transitive: Let $f, g, h \in X$ be functions such that $f R g$ and $g R h$, that is, $f(1) = g(1)$ and $g(1) = h(1)$. Then $f(1) = h(1)$, so $f R h$.

(b) Let $n = 4$. Find all elements in the equivalence class of the function f defined by $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$. By definition, the equivalence class of f is:

$$[f] = \{g \in X \mid g R f\} = \{g \in X \mid g(1) = f(1)\} = \{g \in X \mid g(1) = 2\}.$$

So we need to find all bijective (i.e. one-to-one and onto) functions ~~$f : A \to A$~~ $g : A \to A$ for which $g(1) = 2$. Since $A = \{1, 2, 3\}$ and $g(1) = 2$, we only need to figure out possibilities for $g(2)$ and $g(3)$. The condition that g is bijective implies that $g(2), g(3) \in \{1, 3\}$ since $g(1) = 2$. Again, since g is bijective, this leaves the possibilities that $g(2) = 1$ and $g(3) = 3$, or $g(2) = 3$ and $g(3) = 1$. These are both bijective functions from A to A, and the second is in fact the original function f. Lets call the first h.

So: $[f] = \{f, h\}$ where $h(1) = 2$, $h(2) = 1$, $h(3) = 3$.

8. Prove that if n is an integer for which $5 \nmid n$, then $n^2 \equiv 1 \mod 5$ or $n^2 \equiv 4 \mod 5$.

If n is any integer, then there are 5 possibilities:
$n \equiv 0 \mod 5$, $n \equiv 1 \mod 5$, $n \equiv 2 \mod 5$, $n \equiv 3 \mod 5$, or $n \equiv 4 \mod 5$.
Assuming that $5 \nmid n$, we are left with 4 possibilitier.

Proof by cases:

Case 1 $n \equiv 1 \mod 5$. Then:
$n^2 \equiv 1^2 \mod 5 \equiv 1 \mod 5$.

Case 2 $n \equiv 2 \mod 5$. Then:
$n^2 \equiv 2^2 \mod 5 \equiv 4 \mod 5$.

Case 3 $n \equiv 3 \mod 5$. Then:
$n^2 \equiv 3^2 \mod 5 \equiv 9 \mod 5 \equiv 4 \mod 5$.

Case 4 $n \equiv 4 \mod 5$. Then:
$n^2 \equiv 4^2 \mod 5 \equiv 16 \mod 5 \equiv 1 \mod 5$.

In each case, $n^2 \equiv 1 \mod 5$ or $n^2 \equiv 4 \mod 5$.

9. Let $\mathbb{Z}_9$ be the set of congruence classes of integers modulo 9. Find the subset of $\mathbb{Z}_9^*$ $= \mathbb{Z}_9 - \{[0]_9\}$ consisting of all elements $[a]_9$ for which there exists $[x]_9 \in \mathbb{Z}_9^*$ such that $[a]_9 \cdot_9 [x]_9 = [0]_9$.

$\mathbb{Z}_9^* = \{[1]_9, [2]_9, [3]_9, [4]_9, [5]_9, [6]_9, [7]_9, [8]_9\}$

Examples: $[3]_9 \cdot_9 [3]_9 = [0]_9$

$[6]_9 \cdot_9 [3]_9 = [0]_9$

Conjecture The subset is $\{[3]_9, [6]_9\}$.

Reason why the conjecture is true:

$[a]_9 \cdot_9 [x]_9 = [ax]_9 = [0]_9$ ~~implies~~ iff $9 \mid ax$, i.e.

$ax = 9k$ for some integer k.

Consider the prime factorization of ax; and of a, x:

$ax = 3^2 \cdot k$ where $k = p_1 p_2 \cdots p_r$.

So $3 \mid a$ and $3 \mid x$ since $[a]_9 \neq [0]_9$, $[x]_9 \neq [0]_9$. So $a = 3$ or $a = 6$.

10. Let $X = \{a, b, c, d, e, f\}$. Which of the following are partitions of X?

(a) $\{\{e, f\}, \{a, c, f\}, \{b, d\}\}$ Not a partition since $\{e, f\} \cap \{a, c, f\} \neq \phi$

(b) $\{\{e, f\}, \{a, c\}, \{b, d\}\}$ Yes, this is a partition.

(c) $\{\{a, b, c, d\}, \{e\}, \{f\}\}$ yes, this is a partition.

(d) $\{\{a, c, d\}, \{b, f\}\}$ Not a partition since $e \notin \{a, c, d\}$, $e \notin \{b, f\}$