

**Math 220 Practice for Exam 2**

1. Let  $a_1 = 1$ ,  $a_2 = 9$ , and  $a_{n+1} = 9a_n - 20a_{n-1}$  for all  $n \geq 2$ . Prove that for all positive integers  $n$ ,  $a_n = 5^n - 4^n$ .

2

2. Let  $a_1 = 3$ ,  $a_2 = 5$ , and  $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$  for all  $n \geq 2$ . Prove that for all positive integers  $n$ ,  $3 \leq a_n \leq 5$ .

3. Consider the following two sets:

$$S = \{n \in \mathbb{Z} \mid n = 3x + 6y \text{ for some } x, y \in \mathbb{Z}\},$$

$$T = \{n \in \mathbb{Z} \mid n = 3x + 2y \text{ for some } x, y \in \mathbb{Z}\}$$

(a) Is  $S \subseteq T$ ? Justify your answer.

(b) Is  $T \subseteq S$ ? Justify your answer.

4. Consider the following statement.

P: For all sets  $A$  and  $B$ ,  $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ .

(a) *Just for this part*, let  $A = \{1, 2, 3, 4\}$  and  $B = \{0, 2, 4\}$ . Find the following sets:

$$A \cup B =$$

$$A \cap B =$$

$$A - B =$$

$$B - A =$$

$$(A \cup B) - (A \cap B) =$$

$$(A - B) \cup (B - A) =$$

(b) Draw a Venn diagram to illustrate the statement P in general.

(c) Prove the statement P.

5. (a) [5] Let  $B = \{2, 3, 5, 8\}$  and  $C = \{3, 7\}$ . Find  $B \times C$  (that is, write out all the elements of this set).

(b) Prove that *for all sets*  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$ , then  $A \times C \subseteq B \times C$ .

6. For each positive integer  $i$ , let  $A_i = \left[ -\frac{1}{i}, \frac{i}{i+1} \right]$ . In the following, you need not prove that your answers are correct.

(a) Find  $A_1 \cup A_2$  and  $A_1 \cap A_2$ .

(b) Find  $\bigcup_{i \in \mathbb{Z}^+} A_i$  and  $\bigcap_{i \in \mathbb{Z}^+} A_i$ .

7. Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(n) = \begin{cases} n - 2, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd} \end{cases}$
- (a) Is  $f$  injective? Justify your answer.

- (b) What is the range of  $f$ ? Justify your answer.

8. Let  $A = \{r, s, t\}$ . Let  $f$  and  $g$  be the functions from  $A$  to  $A$  defined by

$$\begin{aligned} f(r) &= t, & f(s) &= r, & f(t) &= s, \\ g(r) &= s, & g(s) &= t, & g(t) &= t. \end{aligned}$$

(a) Find the function  $f \circ g$  (i.e., specify its values on each element of  $A$ ).

(b) Which of the two functions  $f, g$  is invertible? For each that is invertible, find its inverse function (by specifying its values on each element of  $A$ ).



9. For each of the following functions, state whether it is invertible or not. You need not justify your answer.

(a)  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $f(x) = 3x + 2$  for all  $x \in \mathbb{Z}$ .

(b)  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  defined by  $f(x) = 3x + 2$  for all  $x \in \mathbb{Q}$ .

(c)  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  defined by  $f(x) = x^3$  for all  $x \in \mathbb{Q}$ .

(d)  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^3$  for all  $x \in \mathbb{R}$ .