# Math 653 Homework Assignment 10

1. Let $R$ be a commutative ring. Let $I$ be an ideal of $R$, and let $(I)$ be the ideal of $R[x]$ generated by $I$.
   (a) Prove that $R[x]/(I) \cong (R/I)[x]$.
   (b) Prove that if $I$ is a prime ideal of $R$, then $(I)$ is a prime ideal of $R[x]$. (In particular, an element $p \in R$ that is prime in $R$ is also prime as an element of $R[x]$.)

2. Let $R$ be an integral domain and let $S$ be a multiplicative subset of $R$ such that $0 \notin S$. Prove that $S^{-1}R$ is isomorphic to a subring of the field of fractions of $R$.

3. Let $F$ be a field. Prove that $F$ contains a unique smallest subfield $F_0$ that is isomorphic either to $\mathbb{Q}$ or to $\mathbb{F}_p$ for some prime $p$. (Hint: Recall the definition of the characteristic of a ring, and note that the characteristic of a field must either be 0 or prime. Terminology: The field $F_0$ is called the *prime subfield* of $F$.)

4. Let $R$ be a commutative ring containing a prime ideal $P$. Let $S = R - P$.
   (a) Prove that $S$ is a multiplicative set.
   (b) Prove that $S^{-1}R$ has a unique maximal ideal. (Terminology and notation: $S^{-1}R$ is called the *localization* of $R$ at $P$, and is denoted $R_P$. In general, a commutative ring having a unique maximal ideal is called a *local* ring.)

5. Let $R$ be an integral domain. Prove that $R[x, y]$ is not a principal ideal domain.

6. Let $f(x) = x^5 + 6x^4 + 9x^2 - 12 \in \mathbb{Z}[x]$. Use Eisenstein's Criterion to prove that $f$ is irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.

7. Let $F$ be a field.
   (a) Prove that every nonzero element of $F[[x]]$ is of the form $x^k u$ for some integer $k \geq 0$ and unit $u \in F[[x]]$.
   (b) Prove that $F[[x]]$ is a principal ideal domain whose only ideals are 0, $F[[x]]$, and $(x^k)$ for each $k \geq 1$.

8. (a) Prove that $x + 1$ is a unit in $\mathbb{Z}[[x]]$, but is not a unit in $\mathbb{Z}[x]$.
   (b) Prove that $x^2 + 3x + 2$ is irreducible in $\mathbb{Z}[[x]]$, but not in $\mathbb{Z}[x]$.