Math 653 Homework Assignment 2

1. Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1}\rangle$.
   (a) Show that every element of D_{2n} which is not a power of r has order 2.
   (b) Show that D_{2n} is generated by s and sr, each of which has order 2.

2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Let $Q_8 = \langle A, B\rangle$ as a subgroup of $\mathrm{GL}_2(\mathbb{C})$.
   (a) Show that Q_8 is a nonabelian group of order 8 (called the *quaternion group*).
   (b) Prove that $Q_8 \ncong D_8$. (*Hint: See #1(a) above.*)

3. Let $\mathbb{Z}_n^\times = \{\overline{a} \in \mathbb{Z}_n \mid (a, n) = 1\}$ where (a, n) denotes the greatest common divisor of a and n.
   (a) Use the Euclidean Algorithm (p. 11, Thm. 6.5) to show that for each $\overline{a} \in \mathbb{Z}_n^\times$, there exists $\overline{b} \in \mathbb{Z}_n^\times$ such that $\overline{ab} = 1$.
   (b) Use (a) to prove that $\mathbb{Z}_n^\times$ is a group under the operation $\overline{a} \cdot \overline{b} = \overline{ab}$. (Don't forget to check that this operation is well-defined on the equivalence classes modulo n.)

4. Let G be a group and $G_t = \{g \in G \mid |g| \text{ is finite}\}$.
   (a) Prove that if G is abelian, then G_t is a subgroup of G.
   (b) Show that if G is nonabelian, then G_t may not be a subgroup, by verifying the following example: Let $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $G = \mathrm{GL}_2(\mathbb{Q})$. Show that a has order 4, b has order 3, and ab has infinite order.

5. (a) Prove that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic. (*Hint: See p. 11, Thm. 6.5.*)
   (b) Give an example of a subgroup of $\mathbb{Q}$ that is not finitely generated.

6. Prove that if H is a subgroup of index 2 of a group G, then H is normal.

7. Let $H = \{\sigma \in S_5 \mid \sigma(5) = 5\}$. Show that H is a subgroup of S_5. Is H normal in S_5? (Justify your answer.)

8. Let G be a group, $H < G$, and $g \in G$. Prove that gHg^{-1} is a subgroup of G and that $gHg^{-1} \cong H$.