

Math 653 Homework Assignment 3

Recall the definitions from class: If a group G acts on a set Ω , and $x \in \Omega$, the *orbit* $\mathcal{O}_G(x)$ and *stabilizer* $S_G(x)$ are

$$\mathcal{O}_G(x) = \{y \in \Omega \mid y = gx \text{ for some } g \in G\} \quad \text{and} \quad S_G(x) = \{g \in G \mid gx = x \text{ for all } g \in G\}.$$

1. Let A and B be normal subgroups of a group G such that G/A and G/B are both abelian. Prove that $G/A \cap B$ is abelian.
2. Let G be a group and $C(G) = \{g \in G \mid gh = hg \text{ for all } g \in G\}$, the *center* of G .
 - (a) Prove that if $G/C(G)$ is cyclic, then G is abelian.
 - (b) Let p be a prime. Prove that any group of order p^2 is abelian.
3. Let G be a group. For each pair $g, h \in G$, let $[g, h] = g^{-1}h^{-1}gh$, the *commutator* of g and h . Let $[G, G]$ denote the subgroup of G generated by all the commutators $[g, h]$, for $g, h \in G$. (*This is denoted G' in the text.*)
 - (a) Prove that $[G, G]$ is normal in G .
 - (b) Prove that if H is an abelian group and $f : G \rightarrow H$ is any homomorphism, then $[G, G] < \text{Ker} f$. Thus f induces a homomorphism $\bar{f} : G/[G, G] \rightarrow H$.
4. Let G be a finite group acting on a finite set Ω containing at least two elements. Assume that G acts *transitively* on Ω , that is, for any pair $x, y \in \Omega$, there exists $g \in G$ such that $gx = y$.
 - (a) Let $x \in \Omega$. Show that $\mathcal{O}_G(x) = \Omega$, that is the orbit of x is the whole set Ω .
 - (b) Let $x, y \in \Omega$. Prove that there exists $g \in G$ such that $gS_G(x)g^{-1} = S_G(y)$.
 - (c) Let $x \in \Omega$. Prove that $|\Omega| = |G : S_G(x)|$, and therefore $|\Omega|$ divides $|G|$.
5. Let G be a group, and let $\text{Aut}(G)$ denote the set of all automorphisms of G (that is all bijective group homomorphisms from G to G).
 - (a) Prove that $\text{Aut}(G)$ is a group under composition of functions.
 - (b) Find $\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.
 - (c) Show that $\text{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$. (*See Asst. 2, #3 for the notation \mathbb{Z}_n^\times .*)
6. Let G be a group and $H < G$. Let
$$N_G(H) = \{g \in G \mid gHg^{-1} = H\} \quad \text{and} \quad C_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\},$$
both subgroups of G , called the *normalizer* and *centralizer* of H in G , respectively. Prove that both H and $C_G(H)$ are normal in $N_G(H)$, and that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$.
7. Show that A_4 is not simple by exhibiting a proper normal subgroup of A_4 . Why are there not similar proper normal subgroups of A_n for each $n > 4$?
8. Find the centralizers in S_5 of (1234) and of $(123)(45)$. Justify your answers.
9. Prove the alternating version of Cayley's Theorem: Any finite group is isomorphic to a subgroup of A_n for some n .