Math 653 Homework Assignment 5

1. Let G be a group. Suppose that $H, K < G$ with H normal in G, and $H \cap K = \{1\}$. Prove that $HK \cong H \rtimes K$, a semidirect product with respect to some group homomorphism $\theta : K \to \mathrm{Aut}H$. *(Hint: Consider the action of K on H by conjugation.)*

2. Verify that the direct sum $\sum_{i \in I} A_i$ of a family of abelian groups $\{A_i \mid i \in I\}$ is a subgroup of their direct product $\prod_{i \in I} A_i$.

3. Give an example of groups H_1, H_2, K_1, K_2 such that $H_1 \times H_2 \cong K_1 \times K_2$, and $H_i \ncong K_j$ for any i, j.

4. (a) Give an example of an infinite group each of whose nonidentity elements has order 2.
   (b) Give an example of a group G for which $G \cong G \times G$.

5. Let n be a fixed integer. Let F be a free group on some set, and let N be the subgroup generated by the set $\{x^n \mid x \in F\}$. Prove that N is normal in F.

6. Show that S_n is solvable for $n \leq 4$, but that neither S_3 nor S_4 is nilpotent.

7. Prove that a finite group G is solvable if, and only if, each of its composition factors is of prime order.