Math 653 Homework Assignment 6

Assume all rings have multiplicative identities.

1. Find the ascending and descending central series for the quaternion group Q_8.

2. (a) Let G be a group with a normal subgroup N such that N and G/N are both solvable. Prove that G is solvable.
   (b) Does (a) hold if the word "solvable" is replaced by "nilpotent"? Prove or give a counterexample.

3. Prove that the converse of Lagrange's Theorem holds for finite nilpotent groups. That is: Let G be a finite nilpotent group and m a positive integer dividing $|G|$. Then G has a subgroup of order m. (*Hint: Prove it first for a p-group.*)

4. Let I be a nonempty indexing set, and let R_i be a ring for each $i \in I$.
   (a) Prove that the direct product $\prod_{i\in I} R_i$ is a ring under componentwise addition and multiplication.
   (b) What about the direct sum $\sum_{i\in I} R_i$? (This is the subset of $\prod_{i\in I} R_i$ consisting of all functions f for which $f(i) = 0_{R_i}$ for all but finitely many i.) Show that it satisfies all of the ring axioms *except* that it does not have a multiplicative identity if I is infinite.

5. The *center* of a ring R is $C(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$.[1] Prove that $C(R)$ is a subring of R.

6. Let R be a ring such that $\mathbb{C} \subseteq C(R)$. Let $q \in \mathbb{C}^\times$. Suppose $a, b \in R$ such that $ba = qab$. Prove the *q-binomial theorem* (cf. Theorem III.1.6 in the text): For all positive integers n,
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \qquad \text{where} \qquad \binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!},$$
$(j)_q! = (j)_q(j-1)_q \cdots (3)_q(2)_q(1)_q$, $(i)_q = 1 + q + q^2 + \cdots + q^{i-1}$, $(0)_q! = 1$. (Note that if $q = 1$, then $(i)_q = i$. *Hint: First prove the identity for all positive integers n, $1 \le k \le n$:*
$$\binom{n}{k-1}_q + q^k \binom{n}{k}_q = \binom{n+1}{k}_q .)$$

7. Let R be a ring. An element $r \in R$ is *nilpotent* if $r^n = 0$ for some positive integer n.
   (a) Prove that if R is commutative and $r, s \in R$ are nilpotent, then $r + s$ is nilpotent.
   (b) The conclusion in (a) can fail if R is noncommutative: Let $R = M_2(\mathbb{R})$, $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Verify that r and s are both nilpotent, but that $r + s$ is not nilpotent.

[1]The center of R is denoted $Z(R)$ in some books.