Math 653 Homework Assignment 7

1. Note that the ring of polynomials $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$. Let $R = \mathrm{End}_{\mathbb{R}}(\mathbb{R}[x])$, the ring of all linear transformations from $\mathbb{R}[x]$ to $\mathbb{R}[x]$.
   (a) Let $d, i \in R$ be the functions defined by

$$\begin{aligned} d(a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0) &= n a_n x^{n-1} + \cdots + 2a_2 x + a_1, \\ i(a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0) &= \frac{1}{n+1} a_n x^{n+1} + \cdots + \frac{1}{3} a_2 x^3 + \frac{1}{2} a_1 x^2 + a_0 x. \end{aligned}$$

   Show that $di = 1_R$ but $id \neq 1_R$.
   (b) Find an element $f \in R$ such that $df = 0_R$ but $fd \neq 0_R$.

2. Let R be a commutative ring of characteristic p, a prime.
   (a) Prove that $(a+b)^p = a^p + b^p$ for all $a, b \in R$.
   (b) Let $\phi : R \to R$ be the function defined by $\phi(a) = a^p$ for all $a \in R$. Prove that ϕ is a ring homomorphism (called the *Frobenius homomorphism*).

3. Let R be any ring and let $f, g : \mathbb{Q} \to R$ be two ring homomorphisms. Suppose that $f|_{\mathbb{Z}} = g|_{\mathbb{Z}}$ (i.e. $f(z) = g(z)$ for all $z \in \mathbb{Z}$). Prove that $f = g$.

4. Let $R = \mathbb{Z}_2[x]$ and $I = (x^2 + x + 1)$, the ideal generated by $x^2 + x + 1$ (i.e. all elements of the form $(x^2 + x + 1)p(x)$, where $p(x) \in \mathbb{Z}_2[x]$).
   (a) Show that R/I may be identified with the set $\{0, 1, x, x+1\}$. (For example, identify x with its coset $x + I$.)
   (b) Give addition and multiplication tables for R/I. Is R/I a field?

5. Let R be a commutative ring.
   (a) Let N be the set of all nilpotent elements in R. Prove that N is an ideal of R. (*Hint: See Assignment 6, # 7(a).*)
   (b) More generally, let I be any ideal of R. Let

$$\mathrm{Rad}(I) = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}.$$

   Prove that $\mathrm{Rad}(I)$ is an ideal (called the *radical* of I). (In some texts, this is denoted $\sqrt{I}$. Note that N from part (a) is just $\mathrm{Rad}\{0\}$.)

6. Let R be a ring, and let I and J be ideals of R.
   (a) Define $I + J = \{a + b \mid a \in I, b \in J\}$. Prove that $I + J$ is an ideal and that it is the smallest ideal of R containing both I and J.
   (b) Define $IJ = \left\{ \sum_{i=1}^{n} a_i b_i \mid n \text{ is a positive integer}, a_i \in I, b_i \in J \right\}$. Prove that IJ is an ideal of R and that $IJ \subseteq I \cap J$.
   (c) Prove that if R is commutative and $I + J = R$, then $IJ = I \cap J$.
   (d) Give an example of a ring R and ideals I, J such that $IJ \neq I \cap J$.