

Math 653 Homework Assignment 9

1. Let R be a commutative ring, and S a nonempty subset of R such that $0 \notin S$ and $ab \in S$ whenever $a, b \in S$.

- (a) Use Zorn's Lemma to prove that there is an ideal J of R that is maximal with respect to having empty intersection with S .
- (b) Prove that J is a prime ideal.

2. Let $D \in \mathbb{Z} - \{0\}$, D not a square in \mathbb{Z} , and let $\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}$, a subring of \mathbb{R} or \mathbb{C} . Define $\phi : \mathbb{Z}[\sqrt{D}] - \{0\} \rightarrow \mathbb{Z}$ by

$$\phi(a + b\sqrt{D}) = a^2 - b^2D.$$

- (a) Let $\alpha, \beta \in \mathbb{Z}[\sqrt{D}]$. Prove that $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$.
- (b) Prove that $\alpha \in \mathbb{Z}[\sqrt{D}]$ is a unit in $\mathbb{Z}[\sqrt{D}]$ if, and only if, $\phi(\alpha) = \pm 1$.
- (c) Determine the units in $\mathbb{Z}[i]$. (Here $i = \sqrt{-1}$.)
- (d) Let $p \in \mathbb{Z}$ be prime. If $\alpha \in \mathbb{Z}[\sqrt{D}]$ and $\phi(\alpha) = \pm p$, prove that α is irreducible in $\mathbb{Z}[\sqrt{D}]$.
- (e) Show that $1 + i$, $1 - i$, and 3 are irreducible in $\mathbb{Z}[i]$. Conclude that $1 + i$, $1 - i$, and 3 are prime in $\mathbb{Z}[i]$ but that 2 is not a prime in $\mathbb{Z}[i]$. (You may use the result of #3 below, that $\mathbb{Z}[i]$ is a Euclidean domain.)
- (f) Show that $2 + \sqrt{5}i$ is irreducible in $\mathbb{Z}[\sqrt{5}i]$ but is not prime. Conclude that $\mathbb{Z}[\sqrt{5}i]$ is an integral domain that is not a unique factorization domain.

3. Prove that $\mathbb{Z}[i]$ is a Euclidean domain, with norm ϕ given above, as follows.

- (a) Let $\alpha, \beta \in \mathbb{Z}[i]$, $\beta \neq 0$. Show that $\alpha\beta^{-1} = r + si$ for some $r, s \in \mathbb{Q}$.
- (b) If $r, s \in \mathbb{Z}$, then $\alpha = (r + si)\beta$ with $r + si \in \mathbb{Z}[i]$. If not, let p be an integer closest to r and q an integer closest to s . Let $\theta = (r - p) + (s - q)i$ and $\gamma = \theta\beta$. (Note that $\gamma \in \mathbb{Z}[i]$ since $\gamma = \alpha - (p + qi)\beta$.) Show that

$$\phi(\theta) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad \text{and} \quad \phi(\gamma) \leq \frac{1}{2}\phi(\beta).$$

It follows that $\alpha = (p + qi)\beta + \gamma$ with $\gamma = 0$ or $\phi(\gamma) < \phi(\beta)$.

4. Let R be an integral domain, $r \in R$, $r \neq 0$, and let S be the set of all proper principal ideals of R . Prove that r is irreducible if and only if the ideal (r) is maximal in the set S .