

Math 654 Homework Assignment 1

1. Let $V = \mathbb{R}^2$, and $T : V \rightarrow V$ be the linear transformation given by projection onto the y -axis. Then V is an $\mathbb{R}[x]$ -module via $p(x) \cdot v = p(T)(v)$ ($v \in V$), as described in class. Find the $\mathbb{R}[x]$ -submodules of V . Is V a cyclic R -module?
2. Let R be a ring, I an ideal, and M an R -module. Assume that $im = 0$ for all $i \in I$ and $m \in M$. Prove that M can be made into an R/I -module by defining $(r + I)m = rm$ for all $r \in R$, $m \in M$. (That is, show that this action of R/I on M is well-defined, and that M is indeed an R/I -module under this action.)
3. Let R be a ring and M an R -module. The *annihilator* of M in R is
$$\text{Ann}(M) = \{a \in R \mid am = 0 \text{ for all } m \in M\}.$$
 - (a) Prove that $\text{Ann}(M)$ is a (two-sided) ideal of R .
 - (b) Let V be a vector space over a field F , and $T : V \rightarrow V$ a linear transformation. Let V be an $F[x]$ -module as in #1 above. Explain that $\text{Ann}(V)$ is the ideal generated by the minimal polynomial of T .
 - (c) Let G be a finite abelian group (i.e. \mathbb{Z} -module). Explain that $\text{Ann}(G)$ is the ideal generated by $\exp(G)$, the exponent of G .
4. Let R be a ring, M an R -module, and N a submodule of M . If N and M/N are both finitely generated, prove that M is finitely generated.
5. Let $z \in R$ be an element in the center of R , that is, $zr = rz$ for all $r \in R$. Let $zM = \{zm \mid m \in M\}$.
 - (a) Prove that zM is a submodule of M .
 - (b) Let $e \in R$ be a *central idempotent*, that is, e is in the center of R and $e^2 = e$. Prove that $M = eM \oplus (1 - e)M$.
6. Let R be a ring, and M an R -module. The set of *torsion* elements of M is
$$\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$
 - (a) Let G be a finitely generated abelian group (i.e. \mathbb{Z} -module). Describe $\text{Tor}(G)$.
 - (b) If R is an integral domain, prove that $\text{Tor}(M)$ is a submodule of M .
 - (c) Give an example of a ring R and an R -module M such that $\text{Tor}(M)$ is not a submodule. (Hint: Consider the torsion elements in the R -module R .)
7. A nonzero R -module M is *simple* (or *irreducible*) if $\{0\}$ and M are its only submodules.
 - (a) Find all simple \mathbb{Z} -modules.
 - (b) Prove that M is simple if, and only if, M is a cyclic module generated by any of its nonzero elements.
 - (c) Assume R is commutative. Prove that M is irreducible if, and only if, M is isomorphic (as an R -module) to R/I for some maximal ideal I of R .
8. Prove *Schur's Lemma*: Let R be a ring and M and N be two simple R -modules. Then every nonzero element of $\text{Hom}_R(M, N)$ is an isomorphism. In particular, $\text{End}_R(M)$ is a division ring and $\text{Hom}_R(M, N) = 0$ if $M \not\cong N$. (Hint: Consider the kernel and image of each homomorphism.)