Math 654 Homework Assignment 2

Assume R is a *commutative* ring (with 1). If M is an R-module, recall

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

We say M is a *torsion R-module* if $\mathrm{Tor}(M) = M$.

1. Let $M = \oplus_{i \in I} R$, a free R-module with basis $\{e_i\}_{i \in I}$. (This is the notation from class: $e_i = (r_j)$ where $r_j = 1$ if $j = i$ and $r_j = 0$ if $j \neq i$.) If I is *infinite*, show that the dual functions e_i^* $(i \in I)$ do *not* form a basis of M^*, where $M^* = \mathrm{Hom}_R(M, R)$.

2. Let M, M', N, N' be R-modules, and let $f : M \to M'$, $g : N \to N'$ be R-module homomorphisms.
   (a) Show that (left) composition with g induces an R-module homomorphism from $\mathrm{Hom}_R(M, N)$ to $\mathrm{Hom}_R(M, N')$.
   (b) Show that (right) composition with f induces an R-module homomorphism from $\mathrm{Hom}_R(M', N)$ to $\mathrm{Hom}_R(M, N)$.

3. Let A be any abelian group and let m, n be positive integers greater than 1.
   (a) Show that $\mathbb{Z}_m \otimes_{\mathbb{Z}} A \cong A/mA$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m]$ where $A[m]$ is the subgroup $\{a \in A \mid ma = 0\}$ of A.
   (b) Use part (a) to show that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}$ and $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)}$, where (m, n) is the greatest common divisor of m and n.

4. Let $\mathbb{Q}$ denote the (additive) group of rational numbers.
   (a) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.
   (b) Let A be a torsion abelian group (i.e. torsion Z-module). Show that $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

5. Let M, N, U be R-modules. Let $L(M \times N; U)$ denote the set of bilinear maps from $M \times N$ to U, an abelian group under addition of functions. Prove that $L(M \times N; U) \cong \mathrm{Hom}_R(M \otimes_R N, U)$, an isomorphism of abelian groups.

6. Let M and N be R-modules. Prove that $M \otimes_R N \cong N \otimes_R M$.

7. Let M, M', N, N' be R-modules, and let $f : M \to M'$, $g : N \to N'$ be R-module homomorphisms. Prove that there is a unique R-module homomorphism from $M \otimes_R N$ to $M' \otimes_R N'$ (denoted $f \otimes g$) such that $m \otimes n$ is mapped to $f(m) \otimes g(n)$ for all $m \in M, n \in N$.

8. Assume R is an integral domain. The *rank* of an R-module M is the maximal number of linearly independent elements of M.
   (a) Suppose an R-module M has rank n, and $x_1, \ldots, x_n$ is a maximal set of linearly independent elements of M. Prove that the submodule N generated by $x_1, \ldots, x_n$ is isomorphic to R^n and that M/N is a torsion R-module.
   (b) Conversely, suppose an R-module M has a free R-submodule N of rank n such that M/N is a torsion R-module. Prove that M has rank n.
   (c) If M and N are R-modules of ranks m and n, respectively, prove that the rank of $M \oplus N$ is $m + n$. (Hint: Use both parts (a) and (b).)

9. Let $R = \mathbb{Z}[x]$ and $M = (2, x)$ the ideal generated by 2 and x, considered to be an R-submodule of R. Show that M has rank 1 but that M is not *free* of rank 1.