Math 654 Homework Assignment 3

1. Let F be a field and R an algebra over F (not necessarily commutative). Let R^{op} denote the *opposite* algebra to R, that is R as a vector space, and with multiplication $r \cdot_{op} s = sr$ for all $r, s \in R$. Let $R \otimes_F R^{op}$ denote the tensor product algebra: Multiplication is given by $(r \otimes r')(s \otimes s') = rs \otimes s'r'$ for all $r, r', s, s' \in R$. Show that an R-bimodule may be identified with a left $R \otimes_F R^{op}$-module, and vice versa.

2. Let R be a commutative ring and M a free R-module of finite rank. Let $f : M \to M$ be an R-module homomorphism. Prove that $\det(f) \in R^{\times}$ if, and only if, f is invertible.

3. Let G be a finitely generated abelian group. Prove that $G \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r$ for some nonnegative integer r.

4. (a) Find the Jordan canonical form of the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by the matrix

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

   (b) Find a basis of $\mathbb{C}^3$ with respect to which T is represented by the matrix in Jordan canonical form.

5. Determine which of the following matrices are similar:

$$\begin{pmatrix} -1 & 4 & -4 \\ 2 & -1 & 3 \\ 0 & -4 & 3 \end{pmatrix} \quad \begin{pmatrix} -3 & -4 & 0 \\ 2 & 3 & 0 \\ 8 & 8 & 1 \end{pmatrix} \quad \begin{pmatrix} -3 & 2 & -4 \\ 2 & 1 & 0 \\ 3 & -1 & 3 \end{pmatrix} \quad \begin{pmatrix} -1 & 4 & -4 \\ 0 & -3 & 2 \\ 0 & -4 & 3 \end{pmatrix}$$

6. (Rational Canonical Form) Let V be a finite dimensional vector space over a field F, and $T : V \to V$ a linear transformation. Then V is an $F[x]$-module via $p(x) \cdot v = p(T)(v)$ for all $v \in V$, $p(x) \in F[x]$.
   (a) Use the *invariant factors* form of the classification of finitely generated modules over a PID to express V as a direct sum of quotients of $F[x]$.
   (b) Note that T preserves each quotient $F[x]/(a(x))$ from part (a). Find the matrix of the action of T on $F[x]/(a(x))$ with respect to the basis $\overline{1}, \ldots, \overline{x}^{k-1}$, where $\deg(a(x)) = k$ and $\overline{x}^i$ denotes the coset of x^i modulo $(a(x))$. (This matrix should involve the coefficients of $a(x)$ as some of its entries.)
   (c) Combine parts (a) and (b) to find the matrix of T with respect to the basis of V that corresponds to the union of the bases of the various quotients $F[x]/(a(x))$. This is the *rational canonical form* of T.

   (Remark: *Jordan* canonical form, as discussed in class, comes from the *elementary divisors* form of the classification of finitely generated modules over a PID. *Rational* canonical form is so-named because it is not necessary to assume that the eigenvalues of T are in F.)