Math 654 Homework Assignment 4

1. Let $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ be a short exact sequence of R-modules. Prove that it is split if, and only if, there is an R-module homomorphism $\delta : V \to U$ such that $\delta \circ \alpha = \mathrm{id}_U$.

2. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Prove that under the induced maps,
$$0 \to \mathrm{Hom}_R(C, D) \xrightarrow{\beta^*} \mathrm{Hom}_R(B, D) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, D)$$
is exact for any R-module D.

3. Let R be a commutative ring and let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Prove that under the induced maps,
$$A \otimes_R D \xrightarrow{\alpha_*} B \otimes_R D \xrightarrow{\beta_*} C \otimes_R D \to 0$$
is exact for any R-module D. *(Hint: Look in Hungerford.)*

4. Prove that a direct sum $\oplus_{i \in I} P_i$ of R-modules is projective if, and only if, P_i is projective for all $i \in I$.

5. Let R be a principal ideal domain. Prove that every finitely generated projective R-module is free.

6. Let M, N be R-modules.
   (a) Use #2 above to prove $\mathrm{Ext}^0_R(M, N) \cong \mathrm{Hom}_R(M, N)$.
   (b) Assume R is commutative. Use #3 above to prove $\mathrm{Tor}^R_0(M, N) \cong M \otimes_R N$.

For # 7,8, use the projective resolutions we found in class.

7. Let N be a $\mathbb{Z}$-module, and p a positive integer. Show that
$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_p, N) \cong \{n \in N \mid pn = 0\}, \quad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p, N) \cong N/pN,$$
and $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}_p, N) = 0$ for all $i \geq 2$.

8. Let $\mathbb{F}$ be a field and $R = \mathbb{F}[x]/(x^2)$. Consider $\mathbb{F}$ to be an R-module on which x acts as multiplication by 0. Show that $\mathrm{Tor}^R_i(\mathbb{F}, \mathbb{F}) \cong \mathbb{F}$ for all $i \geq 0$.

9. Let $\mathbb{F}$ be a field, n a positive integer, $n \geq 2$, and $R = \mathbb{F}[x]/(x^n)$. Consider $\mathbb{F}$ to be an R-module on which x acts as multiplication by 0.
   (a) Construct a projective resolution of the R-module $\mathbb{F}$, similar to how it was done in class when $n = 2$.
   (b) Use your projective resolution to find $\mathrm{Ext}^i_R(\mathbb{F}, \mathbb{F})$ for each $i \geq 0$.

10. Let R be a ring. Prove that any direct summand of an injective R-module I is injective.