Math 654 Homework Assignment 5

1. Let $p(x) = x^3 - 6x^2 + 9x + 3 \in \mathbb{Q}[x]$, and let α be any root of $p(x)$ in $\mathbb{C}$. Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$, and a basis of $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$. Express α^5 as a linear combination of your basis elements.

2. Let $p(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$, and let α be any root of $p(x)$ in $\mathbb{C}$. Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$, and show that $\sqrt{2} \notin \mathbb{Q}(\alpha)$.

3. Find $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and find a basis of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$. Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

4. Let K/F be a field extension and $\alpha \in K$ such that $[F(\alpha) : F]$ is odd. Prove that $F(\alpha) = F(\alpha^2)$.

5. Let $f(x)$ be an irreducible polynomial of degree n over a field F. Let $g(x) \in F[x]$. Prove that every irreducible factor of $f(g(x))$ has degree divisible by n.

6. Let K_1 and K_2 be finite extensions of a field F, both contained in a field K.
   (a) Prove that $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$.
   (Hint: Let $\alpha_1, \ldots, \alpha_m$ be a basis for K_1 over F, and $\beta_1, \ldots, \beta_n$ be a basis for K_2 over F. Write $K_1 = F(\alpha_1, \ldots, \alpha_m)$ and $K_2 = F(\beta_1, \ldots, \beta_n)$.)
   (b) If $[K_1 : F] = m$ and $[K_2 : F] = n$ with $(n, m) = 1$, prove that $[K_1K_2 : F] = mn$.

7. Determine the splitting field of each of the following polynomials over $\mathbb{Q}$.
   (a) $x^4 + 4$ (b) $x^6 + 1$

8. (a) Let F be an algebraically closed field. Prove that there are no nontrivial finite field extensions of F.
   (b) Prove that every algebraically closed field is infinite.

9. Let F be a field of positive characteristic p. By Math 653, Assignment 7, #2, $(a + b)^p = a^p + b^p$ for all $a, b \in F$, and the map $\phi : F \to F$ defined by $\phi(a) = a^p$ for all $a \in F$ is a ring homomorphism (the *Frobenius map*).
   (a) Prove that ϕ is injective.
   (b) Suppose F is finite, and let $a \in F$. Prove that $x^p - a$ has a root in F.
   (c) Let $\mathbb{F}_p$ be the field of p elements. Prove that $a^p = a$ for all $a \in \mathbb{F}_p$, and that $(f(x))^p = f(x^p)$ for all $f(x) \in \mathbb{F}_p[x]$. (Hint: $|\mathbb{F}_p^\times| = p - 1$.)