Math 654 Homework Assignment 6

1. Prove that $\mathrm{Aut}(\mathbb{R}/\mathbb{Q}) = 1$.

2. Let K be the splitting field of $x^4 - 4$ over $\mathbb{Q}$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$.
   (a) Describe K as an extension of $\mathbb{Q}$ by one or two elements.
   (b) Determine G and find all subgroups of G and their fixed fields.

3. Show that $\mathbb{Q}\left(\sqrt{2+\sqrt{2}}\right)$ is a Galois extension of $\mathbb{Q}$, and determine its Galois group.

4. Let $a, b, c \in \mathbb{Z}$, $a \neq 0$, and let K be the splitting field of $ax^2 + bx + c$ over $\mathbb{Q}$. Determine all possible Galois groups $G = \mathrm{Gal}(K/\mathbb{Q})$, and give conditions on a, b, c under which each group occurs.

5. Let K and L be two Galois extensions of a field F.
   (a) Prove that $K \cap L$ is Galois over F.
   (b) Prove that KL is Galois over F.

6. Let p be a prime. Let K/F be a Galois extension of degree p^n for some positive integer n. Prove that there are Galois extensions of F contained in K of all possible degrees, $1, p, p^2, \ldots, p^{n-1}, p^n$.

7. Prove that every finite group occurs as the Galois group of *some* field extension. (Compare with the *inverse Galois problem* (unsolved), that of determining which finite groups arise as Galois groups of extensions of $\mathbb{Q}$.) (*Hint: Let F be any field and $K = F(x_1, \ldots, x_n)$, the field of rational functions in variables $x_1, \ldots, x_n$. Let $s_1, \ldots, s_n$ denote the elementary symmetric polynomials, and $E = F(s_1, \ldots, s_n)$. Then K/E is Galois with Galois group S_n, the symmetric group of degree n, by a result of Emil Artin. Now use Cayley's Theorem and the Fundamental Theorem of Galois Theory.*)