YETTER-DRINFELD MODULES UNDER COCYCLE TWISTS

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To Susan Montgomery in honor of her distinguished career

Abstract. We give an explicit formula for the correspondence between simple Yetter-Drinfeld modules for certain finite-dimensional pointed Hopf algebras H and those for cocycle twists H^σ of H. This implies an equivalence between modules for their Drinfeld doubles. To illustrate our results, we consider the restricted two-parameter quantum groups $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ under conditions on the parameters guaranteeing that $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is a Drinfeld double of its Borel subalgebra. We determine explicit correspondences between $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules for different values of r and s and provide examples where no such correspondence can exist. Our examples were obtained via the computer algebra system Singular::Plural.

1. Introduction

Radford [22] gave a construction of simple Yetter-Drinfeld modules for a pointed Hopf algebra H, whose group $G(H)$ of grouplike elements is abelian, under fairly general hypotheses on H. These simple modules are in one-to-one correspondence with the Cartesian product of $G(H)$ with its dual group, and are realized as vector subspaces of the Hopf algebra H itself. Radford and Schneider [23] generalized this method to include a much wider class of Hopf algebras given by cocycle twists of tensor product Hopf algebras. Again the simple modules are in one-to-one correspondence with a set of characters, but this time each simple module is realized as the quotient of a Verma module by its unique maximal submodule (equivalently, by the radical of the Shapovalov form) reminiscent of standard Lie-theoretic methods. Thus, the work of Radford and Schneider ties Radford's explicit realization of simple modules as vector subspaces of the Hopf algebra to more traditional methods. One advantage of Radford's approach is that it is purely Hopf-theoretic, and so a priori there are no restrictions on parameters as often occur in Lie theory.

In this paper, we determine how Radford's construction behaves under cocycle twists. We give a precise correspondence between simple Yetter-Drinfeld modules of H and those of the cocycle twist H^σ for a large class of Hopf algebras H (see Theorem 3.9 below). This result uses a theorem of Majid and Oeckl [17] giving a category equivalence between Yetter-Drinfeld modules for H and those for H^σ. We then apply a result of Majid [16] relating Yetter-Drinfeld H-modules for a finite-dimensional Hopf algebra H to modules

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for the Drinfeld double $D(H)$ of H. In this way we obtain an explicit formulation of a category equivalence between $D(H)$-modules and $D(H^\sigma)$-modules.

In Sections 4 and 5, we specialize to the setting of restricted quantum groups. We use Radford's construction to analyze in detail the simple modules for the restricted two-parameter quantum groups $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$; that is, two-parameter versions of the finite-dimensional quantum group $\mathfrak{u}_q(\mathfrak{sl}_n)$, where q is a root of unity (in fact, $\mathfrak{u}_q(\mathfrak{sl}_n)$ is a quotient of $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$). Some conditions on the parameters r and s are known under which a finite-dimensional two-parameter quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is the Drinfeld double of a Borel subalgebra (see e.g. [4, Thm. 4.8] for the case $\mathfrak{g} = \mathfrak{sl}_n$, [12, Thm. 6.2] for the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$, and [5, Ex. 3.13] for the case $s = r^{-1}$). When the conditions hold, $\mathfrak{u}_{r,s}(\mathfrak{g})$-modules correspond to Yetter-Drinfeld modules for the Borel subalgebra, and these in turn are given by Radford's construction. For $n \geq 3$ (under a mild assumption) there is no Hopf algebra isomorphism between $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ and $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$ unless $r = q^{\pm 1}, s = q^{\mp 1}$, (see [12, Thm. 5.5]) On the other hand, computations in Singular::Plural show that the representations of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ and $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$ can be quite similar when the parameters are related in certain ways. We give a precise explanation for this similarity using our results on cocycle twists. In Theorem 4.12, we exploit an explicit cocycle twist that yields an equivalence of categories of Yetter-Drinfeld modules for the respective Borel subalgebras $H_{r,s}$ and $H_{q,q^{-1}}$, and hence an equivalence of the categories of modules for $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ and $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$, under some conditions on the parameters. Cocycle twists different from the ones used here have been shown to give rise to other multi-parameter versions of quantum groups in earlier work by Reshetikhin [24], by Doi [10], and by Chin and Musson [7].

For particular choices of the parameters, however, there is no such cocycle twist, and in that situation the representation theories of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ and of $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$ can be quite different. As an example of this phenomenon, we have used the computer algebra system Singular::Plural to show in Example 5.6 below that for q a primitive fourth root of unity, the dimensions of the simple modules for $\mathfrak{u}_{1,q}(\mathfrak{sl}_3)$ and for $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3)$ differ significantly. For a wide class of such examples, Radford's construction lends itself to computations using Gröbner basis techniques. We further illustrate the construction by briefly explaining how to use Singular::Plural to compute bases and dimensions of *all* simple modules for some of the one- and two-parameter quantum groups $\mathfrak{u}_q(\mathfrak{sl}_3), \mathfrak{u}_{r,s}(\mathfrak{sl}_3)$. Many such computations appeared in the second author's Ph.D. thesis [19].

2. Preliminaries

Drinfeld doubles and Yetter-Drinfeld modules

Let H be a finite-dimensional Hopf algebra over a field $\mathbb{K}$ with coproduct Δ, counit ε, and antipode S. In this paper, we generally assume that $\mathbb{K}$ is algebraically closed of characteristic 0, although this is not needed for the definitions.

The *Drinfeld double of* H, denoted $D(H)$, is the Hopf algebra defined to be

$$D(H) = (H^*)^{\mathrm{coop}} \otimes H$$

as a coalgebra, and the algebra structure is given by

$$(f \otimes a)(f' \otimes b) = \sum f\left(a_{(1)} \rightharpoonup f' \leftharpoonup S^{-1}(a_{(3)})\right) \otimes a_{(2)} b,$$

for all $f, f' \in H^*$ and $a, b \in H$, where $\langle x \rightharpoonup f \mid y\rangle = \langle f \mid yx\rangle$ and $\langle f \leftharpoonup x \mid y\rangle = \langle f \mid xy\rangle$, for all $x, y \in H$ and $f \in H^*$. In these expressions $\langle\ \mid\ \rangle$ is the natural pairing between H and the dual Hopf algebra $H^* = \mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$, and $(H^*)^{\mathrm{coop}}$ is H^* with the opposite coproduct. (In the above equation and throughout the paper, we are adopting Sweedler notation.) With this structure and a suitable counit and antipode, $D(H)$ is a Hopf algebra. (See [18, Defn. 10.3.5] for details.)

For any bialgebra H, a *left-right Yetter-Drinfeld H-module* $(M, \cdot, \delta)$ is a left H-module M (with action denoted $a \cdot m$ for $a \in H$ and $m \in M$) that is also a right H-comodule via $\delta : M \to M \otimes H$, written $\delta(m) = \sum m_{(0)} \otimes m_{(1)}$ for all $m \in M$, such that the following compatibility condition is satisfied:

$$\sum a_{(1)} \cdot m_{(0)} \otimes a_{(2)} m_{(1)} = \sum (a_{(2)} \cdot m)_{(0)} \otimes (a_{(2)} \cdot m)_{(1)} a_{(1)}$$

for all $a \in H$, $m \in M$. There is also a notion of a right-left Yetter-Drinfeld module, which we do not consider here.

In what follows, all references to Yetter-Drinfeld modules are to left-right ones and all modules considered are left modules.

The category of Yetter-Drinfeld modules over a bialgebra H will be denoted by ${}_H\mathcal{YD}^H$ and the isomorphism class of $M \in {}_H\mathcal{YD}^H$ by $[M]$. Yetter-Drinfeld H-modules are $D(H)$-modules and conversely as a result of the following theorem, which is [16, Prop. 2.2] (see also [18, Prop. 10.6.16]).

Theorem 2.1 (Majid [16])**.** *Let H be a finite-dimensional Hopf algebra. The categories of Yetter-Drinfeld H-modules and of $D(H)$-modules may be identified: A Yetter-Drinfeld H-module M is a $D(H)$-module via*

$$(f \otimes a) \cdot m = \sum \langle f \mid (a \cdot m)_{(1)}\rangle (a \cdot m)_{(0)} \tag{2.2}$$

for all $f \in H^*$, $a \in H$, $m \in M$. Conversely, a $D(H)$-module M is a Yetter-Drinfeld H-module via the restriction of M to a left H-module and to a left H^*-module (equivalently, a right H-comodule).

Radford's construction

Radford [22] gave a construction of all simple Yetter-Drinfeld modules for certain graded Hopf algebras. Although the results in [22] are more general, we will state them only under the assumption that $\mathbb{K}$ is an algebraically closed field of characteristic 0.

Lemma 2.3 (Radford [22, Lem. 2])**.** *Let H be a bialgebra over an algebraically closed field $\mathbb{K}$ of characteristic 0, and suppose H^{op} is a Hopf algebra with antipode S^{op}. If $\beta \in \mathrm{Alg}_{\mathbb{K}}(H, \mathbb{K})$, then $H_\beta = (H, \bullet_\beta, \Delta) \in {}_H\mathcal{YD}^H$, where for all x, a in H,*

$$x \bullet_\beta a = \sum \beta(x_{(2)}) x_{(3)} a S^{\mathrm{op}}(x_{(1)}). \tag{2.4}$$

If g belongs to the grouplike elements $G(H)$ of H, then $H \bullet_\beta g$ is a Yetter-Drinfeld submodule of H_β.

Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded Hopf algebra with $H_0 = \mathbb{K}G$ a group algebra, and $H_n = H_{n+1} = \cdots = (0)$ for some $n > 0$. An algebra map $\beta : H \to \mathbb{K}$ is determined by its restriction to G, which is an element in the dual group $\widehat{G} = \mathrm{Hom}\,(G, \mathbb{K}^\times)$ of all group homomorphisms from G to the multiplicative group $\mathbb{K}^{\times}$. In particular, $\beta(H_n) = 0$ for all $n \geq 1$.

Theorem 2.5 (Radford [22, Cor. 1])**.** *Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded Hopf algebra over an algebraically closed field $\mathbb{K}$ of characteristic 0. Suppose that $H_0 = \mathbb{K}G$ for some finite abelian group G and $H_n = H_{n+1} = \cdots = (0)$ for some $n > 0$. Then*

$$(\beta, g) \mapsto [H \bullet_\beta g]$$

is a bijective correspondence between the Cartesian product of sets $\widehat{G} \times G$ and the set of isomorphism classes of simple Yetter-Drinfeld H-modules.

Note that when G is abelian, then for all $g, h \in G$,

$$h \bullet_\beta g = \beta(h) h g h^{-1} = \beta(h) g.$$

3. Cocycle twists

In this section, we first collect definitions and known results about cocycle twists. Then we apply these results in the context of Radford's construction to obtain in Theorem 3.9 below an explicit correspondence of Yetter-Drinfeld modules under twists.

Definitions 3.1. (a) The *convolution product* of two $\mathbb{K}$-linear maps $\sigma, \sigma' : H \otimes H \to \mathbb{K}$ is defined by

$$\sigma\sigma'(a \otimes b) = \sum \sigma(a_{(1)} \otimes b_{(1)}) \sigma'(a_{(2)} \otimes b_{(2)})$$

for all $a, b \in H$.

(b) A $\mathbb{K}$-linear map $\sigma : H \otimes H \to \mathbb{K}$ on a Hopf algebra H is *convolution invertible* if there exists a $\mathbb{K}$-linear map $\sigma^{-1} : H \otimes H \to \mathbb{K}$ such that

$$\sum \sigma(a_{(1)} \otimes b_{(1)}) \sigma^{-1}(a_{(2)} \otimes b_{(2)}) = \varepsilon(ab) = \sum \sigma^{-1}(a_{(1)} \otimes b_{(1)}) \sigma(a_{(2)} \otimes b_{(2)}).$$

(c) A *2-cocycle* on H is a convolution invertible $\mathbb{K}$-linear map $\sigma \colon H \otimes H \to \mathbb{K}$ satisfying

$$\sum \sigma(a_{(1)} \otimes b_{(1)}) \sigma(a_{(2)} b_{(2)} \otimes c) = \sum \sigma(b_{(1)} \otimes c_{(1)}) \sigma(a \otimes b_{(2)} c_{(2)})$$

and $\sigma(a \otimes 1) = \sigma(1 \otimes a) = \varepsilon(a)$, for all $a,\ b,\ c \in H$.

(d) Given a 2-cocycle σ on H, the *cocycle twist* H^σ of H by σ has $H^\sigma = H$ as a coalgebra, and the algebra structure on H^σ is given by

$$a \cdot_\sigma b = \sum \sigma(a_{(1)} \otimes b_{(1)}) a_{(2)} b_{(2)} \sigma^{-1}(a_{(3)} \otimes b_{(3)}) \tag{3.2}$$

for all $a, b \in H^\sigma$. The antipode of H^σ is

$$S^\sigma(a) = \sum \sigma\left(a_{(1)} \otimes S(a_{(2)})\right) S(a_{(3)}) \sigma^{-1}\left(S(a_{(4)}) \otimes a_{(5)}\right), \tag{3.3}$$

where S is the antipode of H.

Recall that similarly a *2-cocycle* on a group G is a map $\sigma\colon G\times G\to\mathbb{K}^\times$ such that for all $g,h,k\in G$,

$$\sigma(g,h)\sigma(gh,k)=\sigma(h,k)\sigma(g,hk). \tag{3.4}$$

A 2-cocycle σ on G is a *2-coboundary* if there exists a function $\gamma : G\to\mathbb{K}^\times$ such that $\sigma = d\gamma$ where for all $g,h\in G$,

$$d\gamma(g,h) := \gamma(g)\gamma(h)\gamma(gh)^{-1}.$$

The cocycle σ is said to be *normalized* if $\sigma(g,1)=1=\sigma(1,g)$ for all $g\in G$.

The set of all (normalized) 2-cocycles on G forms an abelian group under pointwise multiplication, and the 2-coboundaries form a subgroup. The quotient group of 2-cocycles modulo 2-coboundaries is denoted $\mathrm{H}^2(G,\mathbb{K}^\times)$ and is known as the *Schur multiplier* of G.

If G is abelian, $\beta : G\to\mathbb{K}^\times$ is a group homomorphism, and σ is a 2-cocycle on G, then the function $\beta_{g,\sigma} : G\to\mathbb{K}^\times$ given by

$$\beta_{g,\sigma}(h)=\beta(h)\sigma(g,h)\sigma^{-1}(h,g) \tag{3.5}$$

is also a group homomorphism for each $g\in G$; this may be verified directly from the cocycle identity, and it follows as well from the general theory below.

Assume $H=\bigoplus_{n=0}^{\infty}H_n$ is a graded Hopf algebra with $H_0=\mathbb{K}G$, where G is a finite abelian group. A normalized 2-cocycle $\sigma\colon G\times G\to\mathbb{K}^\times$ extends linearly to a Hopf algebra 2-cocycle on the group algebra $\mathbb{K}G$ (we abuse notation and call this extension σ also), where $\sigma(g\otimes h)=\sigma(g,h)$ for g, $h\in G$. Let $\pi\colon H\to\mathbb{K}G$ be the projection onto H_0, and let $\sigma_\pi\colon H\otimes H\to\mathbb{K}$ be defined by $\sigma_\pi=\sigma\circ(\pi\otimes\pi)$. Then σ_π is a Hopf algebra 2-cocycle on H (see for example [1]). The convolution inverse of σ_π is $\sigma_\pi^{-1}=(\sigma^{-1})_\pi$ where $\sigma^{-1}\colon G\times G\to\mathbb{K}^\times$ is given by $\sigma^{-1}(g,h)=\sigma(g,h)^{-1}$ for all $g,h\in G$. By abuse of notation, we also denote σ_π by σ for simplicity.

The resulting cocycle twist H^σ is a graded Hopf algebra with the same grading as H, and $(H^\sigma)_0=\mathbb{K}G$ as a Hopf algebra. Let σ' be another 2-cocycle on G extended to H. Since σ and σ' factor through the projection π to $\mathbb{K}G$ (a cocommutative Hopf algebra), the convolution product $\sigma\sigma'$ is again a 2-cocycle on H. Suppose now that $d\gamma$ is a 2-coboundary coming from the map $\gamma : G\to\mathbb{K}^\times$. Then $H^{(d\gamma)(\sigma)}\cong H^\sigma$ as Hopf algebras: One may check that the map $\psi : H^{(d\gamma)(\sigma)}\to H^\sigma$ defined by

$$\psi(a)=\sum\gamma(a_{(1)})a_{(2)}\gamma(a_{(3)})^{-1}\quad\text{for all } a\in H,$$

is an isomorphism with inverse given by $\psi^{-1}(a)=\sum\gamma(a_{(1)})^{-1}a_{(2)}\gamma(a_{(3)})$, where we assume γ has been extended to H by setting $\gamma(H_n)=0$ for all $n\neq 0$ and by letting γ act linearly on $H_0=\mathbb{K}G$ (compare [9] or [18, Thm. 7.3.4]).

Definition 3.6. If M is Yetter-Drinfeld H-module and σ is a 2-cocycle of H, there is a corresponding Yetter-Drinfeld H^σ-module, denoted M^σ, defined as follows: It is M as a comodule, and the H^σ-action is given by

$$a\cdot^\sigma m=\sum\sigma\left((a_{(2)}\cdot m_{(0)})_{(1)}\otimes a_{(1)}\right)(a_{(2)}\cdot m_{(0)})_{(0)}\sigma^{-1}(a_{(3)}\otimes m_{(1)}),$$

for all $a\in H$, $m\in M$ (see [6, (13)]).

We will use the following equivalence of categories of Yetter-Drinfeld modules. In general, if σ is a 2-cocycle on H, then σ^{-1} is a 2-cocycle on H^σ; this follows from [6, (10)], using the definition of the multiplication on H^σ in (3.2). (In the special case that σ is induced from the group of grouplike elements of H as described above, which is the only case we consider in this paper, σ^{-1} is a 2-cocycle on H as well.) The following result is due originally to Majid and Oeckl [17, Thm. 2.7]; the formulation of it for left-right Yetter-Drinfeld modules can be found in Chen and Zhang [6, Cor. 2.7].

Theorem 3.7. *Let σ be a 2-cocycle on the Hopf algebra H. Then the categories ${}_H\mathcal{YD}^H$ and ${}_{H^\sigma}\mathcal{YD}^{H^\sigma}$ are monoidally equivalent under the functor*

$$F_\sigma\colon {}_H\mathcal{YD}^H \to {}_{H^\sigma}\mathcal{YD}^{H^\sigma} \tag{3.8}$$

which is the identity on homomorphisms, and on the objects is given by $F_\sigma(M) = M^\sigma$. The inverse functor is given by $N \mapsto N^{\sigma^{-1}}$.

In particular, we note that by the definition of the modules M^σ, the category equivalence in the theorem preserves dimensions of modules.

We now apply Theorem 2.5 to obtain both Yetter-Drinfeld H-modules and Yetter-Drinfeld H^σ-modules, under appropriate hypotheses on H. The next theorem gives an explicit description of the correspondence of their simple Yetter-Drinfeld modules.

Theorem 3.9. *Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded Hopf algebra over an algebraically closed field $\mathbb{K}$ of characteristic 0 for which $H_0 = \mathbb{K}G$, G is a finite abelian group, and $H_n = H_{n+1} = \cdots = (0)$ for some $n > 0$. Let $\sigma\colon G \times G \to \mathbb{K}^\times$ be a normalized 2-cocycle on G, extended to a 2-cocycle on H. For each pair $(\beta, g) \in \widehat{G} \times G$, we have*

$$(H\bullet_\beta g)^\sigma \cong H^\sigma \bullet_{\beta_{g,\sigma}} g,$$

where $\beta_{g,\sigma}$ is defined in (3.5). The isomorphism maps $h\bullet_\beta^\sigma g$ to $h\bullet_{\beta_{g,\sigma}} g$ for each h in H.

Proof. Since $H\bullet_\beta g$ is a simple Yetter-Drinfeld H-module by Theorem 2.5, and the inverse of the functor F_σ of (3.8) is the identity on homomorphisms, $(H\bullet_\beta g)^\sigma$ is a simple Yetter-Drinfeld H^σ-module.

Now H^σ is also a graded Hopf algebra with $(H^\sigma)_0 = \mathbb{K}G$ and $(H^\sigma)_n = (H^\sigma)_{n+1} = \cdots = 0$. Thus, isomorphism classes of simple Yetter-Drinfeld H^σ-modules are also in one-to-one correspondence with $\widehat{G} \times G$, and we have unique $\beta' \in \widehat{G}$ and $g' \in G$ such that

$$(H\bullet_\beta g)^\sigma \cong H^\sigma \bullet_{\beta'} g'.$$

Let $\phi\colon (H\bullet_\beta g)^\sigma \to H^\sigma \bullet_{\beta'} g'$ be an isomorphism of Yetter-Drinfeld modules. Since ϕ is a comodule map,

$$(\phi \otimes \mathrm{id}\,)(\Delta(g)) = \Delta(\phi(g)).$$

Applying $\varepsilon \otimes \mathrm{id}$ on both sides, we obtain

$$\varepsilon(\phi(g))g = \phi(g).$$

Because $\phi(g) \neq 0$, this equation implies that $\varepsilon(\phi(g)) \neq 0$, so $g \in \mathrm{Im}\,(\phi)$. However, there is a unique grouplike element in $H^\sigma \bullet_{\beta'} g'$; hence $g = g'$. By multiplying ϕ by $\varepsilon(\phi(g))^{-1}$ if necessary, we may assume $\phi(g) = g$. Note that β' is uniquely determined by its images $\beta'(h)$ for all $h \in G$. Let $h \in G$, then

$$\begin{aligned}
\beta'(h)g &= h\bullet_{\beta'} g = h\bullet_{\beta'}\phi(g) = \phi\left(h\bullet_{\beta}^{\sigma} g\right) \\
&= \phi\left(\sum \sigma\left((h\bullet_\beta g)_{(1)} \otimes h\right)(h\bullet_\beta g)_{(0)}\sigma^{-1}(h\otimes g)\right) \\
&= \phi\left(\sum \sigma\left((\beta(h)g)_{(1)} \otimes h\right)(\beta(h)g)_{(0)}\sigma^{-1}(h\otimes g)\right) \\
&= \phi\left(\sigma(g,h)\beta(h)g\sigma^{-1}(h,g)\right) \\
&= \beta(h)\sigma(g,h)\sigma^{-1}(h,g)g.
\end{aligned}$$

Therefore $\beta'(h) = \beta(h)\sigma(g,h)\sigma^{-1}(h,g) = \beta_{g,\sigma}(h)$, as desired. □

4. Restricted quantum groups

We will apply Theorems 2.5, 3.7, and 3.9 to some restricted quantum groups. We focus on the two-parameter quantum groups $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ in this paper. The same techniques may be used more generally on finite-dimensional two-parameter or multi-parameter quantum groups (such as those for example in [12, 27]).

The quantum groups $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$

Let $\epsilon_1, \dots, \epsilon_n$ be an orthonormal basis of Euclidean space $\mathbb{R}^n$ with inner product $\langle\ ,\ \rangle$. Let $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n\}$ and $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \dots, n-1\}$. Then Φ is a finite root system of type A_{n-1} with base Π of simple roots. Let $r, s \in \mathbb{K}^\times$ be roots of unity with $r \ne s$ and let ℓ be the least common multiple of the orders of r and s. Let q be a primitive ℓth root of unity and y and z be nonnegative integers such that $r = q^y$ and $s = q^z$. The following Hopf algebra, which appeared in [4], is a slight modification of one defined by Takeuchi [26].

Definition 4.1. The algebra $U = U_{r,s}(\mathfrak{sl}_n)$ is the unital associative $\mathbb{K}$-algebra generated by $\{e_j,\ f_j,\ \omega_j^{\pm 1},\ (\omega_j')^{\pm 1},\ \ 1 \le j < n\}$, subject to the following relations.

(R1) The $\omega_i^{\pm 1},\ (\omega_j')^{\pm 1}$ all commute with one another and
$\omega_i\omega_i^{-1} = \omega_j'(\omega_j')^{-1} = 1$,
(R2) $\omega_i e_j = r^{\langle\epsilon_i,\alpha_j\rangle}s^{\langle\epsilon_{i+1},\alpha_j\rangle}e_j\omega_i$ and $\omega_i f_j = r^{-\langle\epsilon_i,\alpha_j\rangle}s^{-\langle\epsilon_{i+1},\alpha_j\rangle}f_j\omega_i$,
(R3) $\omega_i' e_j = r^{\langle\epsilon_{i+1},\alpha_j\rangle}s^{\langle\epsilon_i,\alpha_j\rangle}e_j\omega_i'$ and $\omega_i' f_j = r^{-\langle\epsilon_{i+1},\alpha_j\rangle}s^{-\langle\epsilon_i,\alpha_j\rangle}f_j\omega_i'$,
(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(\omega_i - \omega_i')$,
(R5) $[e_i, e_j] = [f_i, f_j] = 0$ if $|i-j| > 1$,
(R6) $e_i^2 e_{i+1} - (r+s)e_ie_{i+1}e_i + rse_{i+1}e_i^2 = 0$,
$e_ie_{i+1}^2 - (r+s)e_{i+1}e_ie_{i+1} + rse_{i+1}^2e_i = 0$,
(R7) $f_i^2f_{i+1} - (r^{-1}+s^{-1})f_if_{i+1}f_i + r^{-1}s^{-1}f_{i+1}f_i^2 = 0$,
$f_if_{i+1}^2 - (r^{-1}+s^{-1})f_{i+1}f_if_{i+1} + r^{-1}s^{-1}f_{i+1}^2f_i = 0$,

for all $1 \le i, j < n$.

The following coproduct, counit, and antipode give U the structure of a Hopf algebra:

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + \omega_i \otimes e_i, & \Delta(f_i) &= 1 \otimes f_i + f_i \otimes \omega_i', \\ \varepsilon(e_i) &= 0, & \varepsilon(f_i) &= 0, \\ S(e_i) &= -\omega_i^{-1} e_i, & S(f_i) &= -f_i(\omega_i')^{-1}, \end{aligned}$$

and ω_i, ω_i' are grouplike, for all $1 \le i < n$.

Let U^0 be the group algebra generated by all $\omega_i^{\pm 1}$, $(\omega_i')^{\pm 1}$ and let U^+ (respectively, U^-) be the subalgebra of U generated by all e_i (respectively, f_i).

Let

$$\mathcal{E}_{j,j} = e_j \quad \text{and} \quad \mathcal{E}_{i,j} = e_i \mathcal{E}_{i-1,j} - r^{-1} \mathcal{E}_{i-1,j} e_i \quad (i > j),$$

$$\mathcal{F}_{j,j} = f_j \quad \text{and} \quad \mathcal{F}_{i,j} = f_i \mathcal{F}_{i-1,j} - s \mathcal{F}_{i-1,j} f_i \quad (i > j).$$

The algebra U has a triangular decomposition $U \cong U^- \otimes U^0 \otimes U^+$ (as vector spaces), and, as shown in [3, 14], the subalgebras U^+, U^- have monomial Poincaré-Birkhoff-Witt (PBW) bases given respectively by

$$\mathcal{E} := \{\mathcal{E}_{i_1,j_1} \mathcal{E}_{i_2,j_2} \cdots \mathcal{E}_{i_p,j_p} \mid (i_1, j_1) \le (i_2, j_2) \le \cdots \le (i_p, j_p) \text{ lexicographically}\},$$

$$\mathcal{F} := \{\mathcal{F}_{i_1,j_1} \mathcal{F}_{i_2,j_2} \cdots \mathcal{F}_{i_p,j_p} \mid (i_1, j_1) \le (i_2, j_2) \le \cdots \le (i_p, j_p) \text{ lexicographically}\}.$$

In [4] it is proven that all $\mathcal{E}_{i,j}^\ell$, $\mathcal{F}_{i,j}^\ell$, $\omega_i^\ell - 1$, and $(\omega_i')^\ell - 1$ $(1 \le j \le i < n)$ are central in $U_{r,s}(\mathfrak{sl}_n)$. The ideal I_n generated by these elements is a Hopf ideal [4, Thm. 2.17], and so the quotient

$$\mathfrak{u}_{r,s}(\mathfrak{sl}_n) = U_{r,s}(\mathfrak{sl}_n)/I_n \tag{4.2}$$

is a Hopf algebra, called the *restricted two-parameter quantum group*. Examination of the PBW bases $\mathcal{E}$ and $\mathcal{F}$ shows that $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is finite dimensional, and in fact, it is a pointed Hopf algebra [4, Prop. 3.2].

Let $\mathcal{E}_\ell$ and $\mathcal{F}_\ell$ denote the sets of monomials in $\mathcal{E}$ and $\mathcal{F}$ respectively, in which each $\mathcal{E}_{i,j}$ or $\mathcal{F}_{i,j}$ appears as a factor at most $\ell - 1$ times. Identifying cosets in $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ with their representatives, we may assume $\mathcal{E}_\ell$ and $\mathcal{F}_\ell$ are bases for the subalgebras of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ generated by the elements e_i and f_i respectively.

Definition 4.3. Let $\mathfrak{b}_{r,s}$ denote the Hopf subalgebra of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ generated by all ω_i, e_i, and let $\mathfrak{b}_{r,s}'$ denote the Hopf subalgebra generated by all ω_i', f_i $(1 \le i < n)$.

Under certain conditions on the parameters r and s, stated explicitly in the next theorem, $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ is a Drinfeld double. The statement of the result differs somewhat from that given in [4] due to the use of a slightly different definition of the Drinfeld double; details on these differences can be found in [20].

Theorem 4.4 ([4, Thm. 4.8]). *Assume $r = q^y$ and $s = q^z$, where q is a primitive ℓth root of unity, and*

$$\gcd(y^{n-1} - y^{n-2}z + \cdots + (-1)^{n-1} z^{n-1}, \ell) = 1. \tag{4.5}$$

Then there is an isomorphism of Hopf algebras $\mathfrak{u}_{r,s}(\mathfrak{sl}_n) \cong (D((\mathfrak{b}_{r,s}')^{\mathrm{coop}}))^{\mathrm{coop}}$.

For simplicity we set

$$H_{r,s} := (\mathfrak{b}'_{r,s})^{\mathrm{coop}} \quad \text{and} \quad G = G(H_{r,s}) = \langle \omega'_i \mid 1 \leq i < n \rangle. \tag{4.6}$$

Then $H_{r,s}$ is a graded Hopf algebra with $\omega'_i \in (H_{r,s})_0$, and $f_i \in (H_{r,s})_1$ for all $1 \leq i < n$. When the conditions of Theorem 4.4 are satisfied, we may apply Theorems 2.1 and 2.5 to obtain $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules as Yetter-Drinfeld $H_{r,s}$-modules via Radford's construction as follows. Note that algebra maps from $H_{r,s}$ to $\mathbb{K}$ are the grouplike elements of $H^*_{r,s}$.

Theorem 4.7. *Assume (4.5) holds. Isomorphism classes of simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules (equivalently, simple Yetter-Drinfeld $H_{r,s}$-modules) are in one-to-one correspondence with $G(H^*_{r,s}) \times G(H_{r,s})$.*

2-cocycles on Borel subalgebras

The group $G = G(H_{r,s})$ is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$. We wish to determine all cocycle twists of $H_{r,s}$ arising from cocycles of G as described in Section 3. By Theorems 2.1 and 3.7, the categories of modules of the Drinfeld doubles of all such cocycle twists are equivalent via an equivalence that preserves comodule structures.

The cohomology group $\mathrm{H}^2(G, \mathbb{K}^\times)$ is known from the following result of Schur [25], which can also be found in [13, Prop. 4.1.3].

Theorem 4.8 (Schur [25])**.** $\mathrm{H}^2((\mathbb{Z}/\ell\mathbb{Z})^m, \mathbb{K}^\times) \cong (\mathbb{Z}/\ell\mathbb{Z})^{\binom{m}{2}}$.

The isomorphism in the theorem is given by induction, using [13, Thm. 2.3.13]:

$$\begin{aligned} \mathrm{H}^2((\mathbb{Z}/\ell\mathbb{Z})^i, \mathbb{K}^\times) &\cong \mathrm{H}^2((\mathbb{Z}/\ell\mathbb{Z})^{i-1}, \mathbb{K}^\times) \times \mathrm{Hom}((\mathbb{Z}/\ell\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/\ell\mathbb{Z})^{i-1}, \mathbb{K}^\times) \\ &\cong \mathrm{H}^2((\mathbb{Z}/\ell\mathbb{Z})^{i-1}, \mathbb{K}^\times) \times \mathrm{Hom}((\mathbb{Z}/\ell\mathbb{Z})^{i-1}, \mathbb{K}^\times), \end{aligned}$$

for $i \geq 2$, which uses the fact that $(\mathbb{Z}/\ell\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/\ell\mathbb{Z})^{i-1} \cong (\mathbb{Z}/\ell\mathbb{Z})^{i-1}$. This isomorphism is given as follows: Identify $(\mathbb{Z}/\ell\mathbb{Z})^i$ with $(\mathbb{Z}/\ell\mathbb{Z}) \times (\mathbb{Z}/\ell\mathbb{Z})^{i-1}$. Let $\psi : (\mathbb{Z}/\ell\mathbb{Z})^{i-1} \times (\mathbb{Z}/\ell\mathbb{Z})^{i-1} \to \mathbb{K}^\times$ be a 2-cocycle, and $\phi : (\mathbb{Z}/\ell\mathbb{Z})^{i-1} \to \mathbb{K}^\times$ be a group homomorphism. The corresponding 2-cocycle $\sigma : (\mathbb{Z}/\ell\mathbb{Z})^i \times (\mathbb{Z}/\ell\mathbb{Z})^i \to \mathbb{K}^\times$ is

$$\sigma((g, h), (g', h')) = \psi(h, h')\phi(g \cdot h')$$

for all $g, g' \in \mathbb{Z}/\ell\mathbb{Z}$ and $h, h' \in (\mathbb{Z}/\ell\mathbb{Z})^{i-1}$, where $g \cdot h'$ is the image of $g \otimes h'$ under the isomorphism $(\mathbb{Z}/\ell\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/\ell\mathbb{Z})^{i-1} \cong (\mathbb{Z}/\ell\mathbb{Z})^{i-1}$, i.e. it is the action of $g \in \mathbb{Z}/\ell\mathbb{Z}$ on the element h' in the $(\mathbb{Z}/\ell\mathbb{Z})$-module $(\mathbb{Z}/\ell\mathbb{Z})^{i-1}$. The asymmetry in the formula is due to the fact that $\mathrm{H}^2(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{K}^\times) = 0$.

Example 4.9. If $i = 2$, and the generators for $(\mathbb{Z}/\ell\mathbb{Z})^2$ are g_1 and g_2, we obtain

$$\sigma(g_1^{i_1} g_2^{i_2}, g_1^{j_1} g_2^{j_2}) = \psi(g_2^{i_2}, g_2^{j_2})\phi(g_1^{i_1} \cdot g_2^{j_2}) = \psi(g_2^{i_2}, g_2^{j_2})\phi(g_2^{i_1 j_2}).$$

Since all 2-cocycles on the cyclic group $\mathbb{Z}/\ell\mathbb{Z}$ are coboundaries, we may assume without loss of generality that ψ is trivial, so that σ is given by the group homomorphism $\phi : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{K}^\times$. If q is a primitive ℓth root of unity, any such group homomorphism just sends the generator of $\mathbb{Z}/\ell\mathbb{Z}$ to a power of q, and thus we obtain representative cocycles, one for each $a \in \{0, 1, \ldots, \ell - 1\}$:

$$\sigma(g_1^{i_1} g_2^{i_2}, g_1^{j_1} g_2^{j_2}) = q^{a i_1 j_2}.$$

Similarly, by induction, we have the following.

Proposition 4.10. *Let q be a primitive ℓth root of unity. A set of cocycles $\sigma : (\mathbb{Z}/\ell\mathbb{Z})^m \to \mathbb{K}^\times$ which represents all elements of $\mathrm{H}^2((\mathbb{Z}/\ell\mathbb{Z})^m, \mathbb{K}^\times)$ is given by*

$$\sigma((g_1^{i_1}\cdots g_m^{i_m},\ g_1^{j_1}\cdots g_m^{j_m})) = q^{\left(\sum_{1\le k<l\le m} a_{k,l}\, i_k j_l\right)} \tag{4.11}$$

where $0 \le a_{k,l} \le \ell - 1$. Thus the cocycles are parametrized by $m \times m$ strictly upper triangular matrices with entries in $\mathbb{Z}/\ell\mathbb{Z}$.

Next we determine isomorphisms among cocycle twists of the Hopf algebras $H_{r,s}$. We will denote by $|r|$ the order of r as a root of unity, and use $\mathrm{lcm}(|r|, |s|)$ to denote the least common multiple of the orders of r and s. Fix $n \ge 2$, and let $G = \langle \omega_i' \mid 1 \le i < n\rangle$, the grouplike elements of $H_{r,s} = (\mathfrak{b}'_{r,s})^{\mathrm{coop}} \subset \mathfrak{u}_{r,s}(\mathfrak{sl}_n)^{\mathrm{coop}}$.

Theorem 4.12. *The following are equivalent:*

(i) *There is a cocycle σ induced from G such that $H^\sigma_{r,s} \cong H_{r',s'}$ as graded Hopf algebras.*

(ii) $\mathrm{lcm}(|r|, |s|) = \mathrm{lcm}(|r'|, |s'|)$ *and* $r'(s')^{-1} = rs^{-1}$.

We note that if $\mathrm{lcm}(|r|, |s|) \ne \mathrm{lcm}(|r'|, |s'|)$, then the dimensions of $H_{r,s}$ and of $H_{r',s'}$ are different.

Proof. First assume that $\mathrm{lcm}(|r|, |s|) = \mathrm{lcm}(|r'|, |s'|)$ and $r'(s')^{-1} = rs^{-1}$. Let $\sigma = \sigma_{r,r'}$ be the cocycle defined by

$$\sigma(\omega_1'^{i_1}\cdots\omega_{n-1}'^{i_{n-1}}, \omega_1'^{j_1}\cdots\omega_{n-1}'^{j_{n-1}}) = (r(r')^{-1})^{\sum_{k=1}^{n-2} i_k j_{k+1}}. \tag{4.13}$$

To compare this with the expression in (4.11), write $r(r')^{-1} = q^a$ for some a, and note that this cocycle coincides with the choice $a_{k,l} = a\delta_{l,k+1}$. We claim that $H^\sigma_{r,s} \cong H_{r',s'}$. We will prove this by verifying relations (R3) and (R7) of Definition 4.1, as well as the nilpotency of certain PBW basis elements. Each of the other relations is either automatic or does not apply to $H_{r,s}$. First we check that the Serre relations (R7) for $H_{r',s'}$ hold in $H^\sigma_{r,s}$:

$$\begin{aligned}
&f_i \cdot_\sigma f_i \cdot_\sigma f_{i+1} - \big((r')^{-1} + (s')^{-1}\big) f_i \cdot_\sigma f_{i+1} \cdot_\sigma f_i + (r')^{-1}(s')^{-1} f_{i+1} \cdot_\sigma f_i \cdot_\sigma f_i \\
&\quad = \sigma(\omega_i', \omega_i')\sigma((\omega_i')^2, \omega_{i+1}) f_i^2 f_{i+1} - ((r')^{-1} + (s')^{-1})\sigma(\omega_i', \omega_{i+1}')\sigma(\omega_i'\omega_{i+1}', \omega_i') f_i f_{i+1} f_i \\
&\qquad\qquad + (r')^{-1}(s')^{-1}\sigma(\omega_{i+1}', \omega_i')\sigma(\omega_i'\omega_{i+1}', \omega_i') f_{i+1} f_i^2 \\
&\quad = (r(r')^{-1})^2 f_i^2 f_{i+1} - ((r')^{-1} + (s')^{-1})(r(r')^{-1}) f_i f_{i+1} f_i + (r')^{-1}(s')^{-1} f_{i+1} f_i^2 \\
&\quad = (r(r')^{-1})^2 \big(f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1}s^{-1} f_{i+1} f_i^2\big) \\
&\quad = 0,
\end{aligned}$$

by (R7) for $H_{r,s}$. Similarly, the other Serre relation in (R7) holds.

Next let

$$\mathcal{F}^\sigma_{i,j} = f_i \cdot_\sigma \mathcal{F}^\sigma_{i-1,j} - s'\mathcal{F}^\sigma_{i-1,j} \cdot_\sigma f_i$$

for $i > j$ and set $\mathcal{F}^\sigma_{i,i} = f_i$. We need to show that $(\mathcal{F}^\sigma_{i,j})^\ell = 0$ in $H^\sigma_{r,s}$ (and that no smaller power of $\mathcal{F}^\sigma_{i,j}$ is zero). In what follows we show by induction on $i - j$ that $\mathcal{F}^\sigma_{i,j} = \mathcal{F}_{i,j}$. Let $\omega'_{i,j} = \omega'_j \cdots \omega'_i$.

- If $i = j$, then $\mathcal{F}^\sigma_{i,i} = f_i = \mathcal{F}_{i,i}$.

- Assume $\mathfrak{F}^\sigma_{i-1,j} = \mathfrak{F}_{i-1,j}$, then

$$\begin{aligned}
\mathfrak{F}^\sigma_{i,j} &= f_i \cdot_\sigma \mathfrak{F}^\sigma_{i-1,j} - s'\mathfrak{F}^\sigma_{i-1,j} \cdot_\sigma f_i \\
&= f_i \cdot_\sigma \mathfrak{F}_{i-1,j} - s'\mathfrak{F}_{i-1,j} \cdot_\sigma f_i \\
&= \sigma(\omega'_i, \omega'_{i-1,j}) f_i \mathfrak{F}_{i-1,j} - s'\sigma(\omega'_{i-1,j}, \omega'_i)\mathfrak{F}_{i-1,j} f_i \\
&= f_i \mathfrak{F}_{i-1,j} - s' r (r')^{-1} \mathfrak{F}_{i-1,j} f_i \\
&= f_i \mathfrak{F}_{i-1,j} - s\mathfrak{F}_{i-1,j} f_i \quad = \quad \mathfrak{F}_{i,j}.
\end{aligned}$$

We will use the following identity which is a consequence of [4, Lem. 2.22]:

$$(\pi \otimes \mathrm{id} \otimes \pi)\Delta^2(\mathfrak{F}_{i,j}) = \omega'_{i,j} \otimes \mathfrak{F}_{i,j} \otimes 1,$$

where π is the projection onto $\mathbb{K}G$ and $G = \langle \omega'_i \mid 1 \leq i < n \rangle$. Then

$$\mathfrak{F}_{i,j} \cdot_\sigma \mathfrak{F}_{i,j} = \sigma(\omega'_{i,j}, \omega'_{i,j})\mathfrak{F}^2_{i,j}$$

and $\underbrace{\mathfrak{F}_{i,j} \cdot_\sigma \cdots \cdot_\sigma \mathfrak{F}_{i,j}}_{m} = \left(\sigma(\omega'_{i,j}, \omega'_{i,j})\right)^{\binom{m}{2}} \mathfrak{F}^m_{i,j} = 0$ if and only if $\mathfrak{F}^m_{i,j} = 0$.

Finally we check that the second relation in (R3) of Definition 4.1 for $H_{r',s'}$ holds in $H^\sigma_{r,s}$. Note that the second relation in (R3) for $H_{r,s}$ can be written as

$$\omega'_i f_j = b_{i,j}\, f_j \omega'_i, \quad \text{where} \quad b_{i,j} = \begin{cases} s, & j = i-1 \\ rs^{-1}, & j = i \\ r^{-1}, & j = i+1 \\ 1, & \text{otherwise.} \end{cases}$$

Hence, we need to verify that $\omega'_i \cdot_\sigma f_j = c_{i,j}\, f_j \cdot_\sigma \omega'_i$, where $c_{i,j} = \begin{cases} s', & j = i-1 \\ r'(s')^{-1}, & j = i \\ (r')^{-1}, & j = i+1 \\ 1, & \text{otherwise;} \end{cases}$

$$\begin{aligned}
\omega'_i \cdot_\sigma f_j = \sigma(\omega'_i, \omega'_j)\omega'_i f_j \quad &= \quad \begin{cases} r(r')^{-1}\omega'_i f_j, & j = i+1 \\ \omega'_i f_j, & \text{otherwise} \end{cases} \\
&= \quad \begin{cases} b_{i,j} r (r')^{-1} f_j \omega'_i, & j = i+1 \\ b_{i,j} f_j \omega'_i, & \text{otherwise} \end{cases} = \begin{cases} s\, f_j \omega'_i, & j = i-1 \\ rs^{-1} f_j \omega'_i & j = i \\ (r')^{-1} f_j \omega'_i & j = i+1 \\ f_j \omega'_i & \text{otherwise.} \end{cases}
\end{aligned}$$

On the other hand

$$
\begin{aligned}
c_{i,j}\, f_j \cdot_\sigma w_i' &= c_{i,j}\sigma(\omega_j',\omega_i')\, f_j\omega_i' = \begin{cases} r(r')^{-1}c_{i,j}f_j\omega_i', & j=i-1\\ c_{i,j}f_j\omega_i', & \text{otherwise}\end{cases}\\
&= \begin{cases} s\, f_j\omega_i', & j=i-1\\ rs^{-1}f_j\omega_i', & j=i\\ (r')^{-1}f_j\omega_i', & j=i+1\\ f_j\omega_i' & \text{otherwise.}\end{cases}
\end{aligned}
$$

Thus $H^\sigma_{r,s}\cong H_{r',s'}$.

Now assume that $H^\sigma_{r,s}\cong H_{r',s'}$ as graded Hopf algebras for some cocycle σ induced from G. Then the dimensions of these two algebras are the same, and so $\mathrm{lcm}(|r|,|s|)=\mathrm{lcm}(|r'|,|s'|)$. Since cohomologous cocycles give isomorphic twists, we may assume without loss of generality that σ is given by formula (4.11). Since $\deg(f_1)=1$, the element f_1 gets mapped under the isomorphism $H^\sigma_{r,s}\xrightarrow{\sim}H_{r',s'}$ to another skew-primitive element of degree 1, which must be a scalar multiple of some f_i. This forces ω_1' to be mapped to ω_i'. Then the second relation in (R3) of Definition 4.1 for $H^\sigma_{r,s}$ is $\omega_1'\cdot_\sigma f_1=\sigma(\omega_1',\omega_1')\omega_1'f_1=rs^{-1}f_1\omega_1'=rs^{-1}f_1\cdot_\sigma\omega_1'$, while the second relation in (R3) for $H_{r',s'}$ is $\omega_i'f_i=r'(s')^{-1}f_i\omega_i'$. The existence of the isomorphism now implies that $r'(s')^{-1}=rs^{-1}$. □

Remark 4.14. The cocycles σ in (4.13) do not exhaust all possible cocycles, as can be seen by comparing (4.13) with Proposition 4.10. Other cocycles will take $H_{r,s}$ to multi-parameter versions of the Borel subalgebra not considered here. For comparison, examples of some different finite-dimensional multi-parameter quantum groups of $\mathfrak{sl}_n$-type with different Borel subalgebras appear in [27].

Remark 4.15. The correspondence between simple Yetter-Drinfeld modules resulting from Theorem 4.12 can be realized explicitly as follows. Assume that $\mathrm{lcm}(|r|,|s|)=\mathrm{lcm}(|r'|,|s'|)$ and $r'(s')^{-1}=rs^{-1}$. Let y,y' be such that $r=q^y$, $r'=q^{y'}$. Recall that by Radford's result (Theorem 2.5), the simple Yetter-Drinfeld modules for $H=H_{r,s}$ have the form $H\bullet_\beta g$ where $g\in G$, and $\beta\in\widehat{G}$. Let $g=(\omega_1')^{d_1}\cdots(\omega_{n-1}')^{d_{n-1}}$ and suppose $\beta\in\widehat{G}$ is defined by integers β_i for which $\beta(\omega_i')=q^{\beta_i}$. Then by Theorem 3.9, the correspondence is given by $(H_{r,s}\bullet_\beta g)^\sigma=H_{r',s'}\bullet_{\beta_{g,\sigma}}g$, where

$$
\beta_{g,\sigma}(\omega_i')=q^{\beta_i}(r(r')^{-1})^{d_{i-1}}(r(r')^{-1})^{-d_{i+1}}=q^{\beta_i+(d_{i-1}-d_{i+1})(y-y')} \tag{4.16}
$$

and $d_0=d_n=0$.

Corollary 4.17. *Let q be a primitive ℓth root of unity. Assume r,s are powers of q.*

(a) *If $|rs^{-1}|=\mathrm{lcm}(|r|,|s|)$, then there is a one-to-one correspondence between simple Yetter-Drinfeld modules for $H_{r,s}$ and those for $H_{1,r^{-1}s}$.*

(b) *If ℓ is a prime and $r\neq s$, there is a one-to-one correspondence between simple Yetter-Drinfeld modules for $H_{r,s}$ and those for $H_{1,r^{-1}s}$. Hence, the simple Yetter-Drinfeld modules of $H_{r,s}$ and $H_{r',s'}$ (and also the simple modules for $D(H_{r,s})$ and $D(H_{r',s'})$) are in bijection for any two pairs (r,s), (r',s'), with $r\neq s$ and*

$r' \neq s'$. In particular, when (4.5) holds for both pairs, there is a one-to-one correspondence between simple modules for $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ and those for $\mathfrak{u}_{r',s'}(\mathfrak{sl}_n)$.

Proof. The proof of Theorem 4.12 shows that when $\mathrm{lcm}(|r|,|s|) = \mathrm{lcm}(|r'|,|s'|)$ and $r'(s')^{-1} = rs^{-1}$, then $H^{\sigma}_{r,s} = H_{\xi r,\xi s} = H_{r',s'}$, where $\xi = r'r^{-1}$ and $\sigma = \sigma_{r,r'}$. Applying this to the case $r' = 1$, $s' = r^{-1}s$, $\xi = r^{-1}$, we get $H^{\sigma}_{r,s} = H_{1,r^{-1}s}$. The correspondence between modules then comes from Theorem 3.9 as in Remark 4.15. The first statement in part (b) follows immediately from (a) since $|r^{-1}s| = \mathrm{lcm}(|r|,|s|) = \ell$. Now observe that the representation theory of $H_{1,q}$ for any primitive ℓth root q of 1 is the same as that of any other. □

If rs^{-1} is a primitive ℓth root of unity for ℓ odd, we may assume $rs^{-1} = q^2$, where q is a primitive ℓth root of unity. Then if $\mathrm{lcm}(|r|,|s|) = \ell$, the algebra $H_{q,q^{-1}}$ is a cocycle twist of $H_{r,s}$. Combined with Theorem 4.4, this implies the following:

Theorem 4.18. *Let ℓ be odd and suppose $\mathrm{lcm}(|r|,|s|) = \ell = |rs^{-1}|$, and q is a primitive ℓth root of 1 with $rs^{-1} = q^2$. Assume that hypothesis (4.5) holds, where $r = q^y$ and $s = q^z$. Then the categories of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules, $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$-modules, and $\mathfrak{u}_{1,r^{-1}s}(\mathfrak{sl}_n)$-modules are all monoidally equivalent.*

Remark 4.19. The one-parameter quantum group $\mathfrak{u}_q(\mathfrak{sl}_n)$ is the quotient of $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$ by the ideal generated by all $\omega_i' - \omega_i^{-1}$. Under the hypotheses of the theorem, each simple $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$-module factors uniquely as a tensor product of a one-dimensional $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$-module and a simple module for $\mathfrak{u}_q(\mathfrak{sl}_n)$, as long as $\gcd(n,\ell) = 1$ by [20, Thm. 2.12]. The converse holds in general: Any $\mathfrak{u}_q(\mathfrak{sl}_n)$-module becomes a $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$-module via the quotient map $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n) \to \mathfrak{u}_q(\mathfrak{sl}_n)$, and we may take the tensor product of a simple $\mathfrak{u}_q(\mathfrak{sl}_n)$-module with any one-dimensional $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$-module to get a simple $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$-module. The one-dimensional modules are described in the next section.

5. Computations

Throughout this section we assume $r = q^y, s = q^z$, where q is a primitive ℓth root of unity and $\mathrm{lcm}(|r|,|s|) = \ell$. We begin by computing all the one-dimensional Yetter-Drinfeld modules for $H_{r,s}$. When (4.5) is satisfied, this gives the one-dimensional modules for $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$. Then we specialize to the case $n = 3$. In [19, 21], the computer algebra system Singular::Plural [11] was used to construct simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules for many values of r and s. Here we give a brief discussion of the calculations from the point of view of the results of this paper and present some of the examples. The examples illustrate how widely different the dimensions of the simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules can be when the Borel subalgebras are not related by cocycle twists.

The one-dimensional Yetter-Drinfeld modules for $H = H_{r,s}$ have the form $H \bullet_\beta g$ for some $g \in G = \langle \omega_i' \mid 1 \le i < n \rangle$. Combining the definition of the $\bullet_\beta$ action in (2.4) with the coproduct formulas in H, we have for all $x \in H$ and $g \in G(H)$,

$$f_i \bullet_\beta x = xS(f_i) + \beta(\omega_i') f_i x (\omega_i')^{-1} = -x f_i (\omega_i')^{-1} + \beta(\omega_i') f_i x (\omega_i')^{-1}, \tag{5.1}$$

and

$$\omega_i' \bullet_\beta g = \beta(\omega_i') \omega_i' g (\omega_i')^{-1} = \beta(\omega_i') g. \tag{5.2}$$

Since $H \bullet_\beta g$ is one-dimensional, $H \bullet_\beta g = \mathbb{K}g$. Now if $g = (\omega_1')^{d_1} \cdots (\omega_{n-1}')^{d_{n-1}}$, then

$$g f_i = r^{d_i - d_{i-1}} s^{d_{i+1} - d_i} f_i g,$$

where $d_0 = 0 = d_n$. Thus, equation (5.1) for $x = g$ implies that

$$f_i \bullet_\beta g = -r^{d_i - d_{i-1}} s^{d_{i+1} - d_i} f_i g (\omega_i')^{-1} + \beta(\omega_i') f_i g (\omega_i')^{-1} \qquad \text{for } 1 \le i < n. \tag{5.3}$$

Since by the PBW basis theorem, the right side is a multiple of g only if it is zero, we obtain that

$$\beta(\omega_i') = r^{d_i - d_{i-1}} s^{d_{i+1} - d_i} \qquad \text{for } 1 \le i < n. \tag{5.4}$$

Hence, each module of the form $H \bullet_\beta g$ which is one-dimensional is determined by g. Conversely, every $g = (\omega_1')^{d_1} \cdots (\omega_{n-1}')^{d_{n-1}} \in G$ determines a one-dimensional module $H \bullet_\beta g$ where β is given by (5.4). Therefore, the number of nonisomorphic one-dimensional modules is ℓ^{n-1}. In summary we have the following result, due originally to the second author (compare [20, Prop. 2.10]).

Proposition 5.5. *Assume $r = q^y$, $s = q^z$, where q is a primitive ℓth root of unity and* $\operatorname{lcm}(|r|, |s|) = \ell$. *Then the one-dimensional Yetter-Drinfeld modules for $H_{r,s} := (\mathfrak{b}_{r,s}')^{\mathrm{coop}}$ are in bijection with the elements of $G = \langle \omega_i' \mid 1 \le i < n \rangle$, under the correspondence taking $g = (\omega_1')^{d_1} \cdots (\omega_{n-1}')^{d_{n-1}}$ to the one-dimensional module $H \bullet_\beta g = \mathbb{K}g$, where β is given by (5.4). Thus, when (4.5) holds, the one-dimensional modules for $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ are in bijection with the elements of G.*

Techniques used in Singular::Plural

We now discuss computer calculations done for small values of ℓ and $n = 3$. In order to use Singular::Plural, the algebras must be given by generators and relations of a particular form which allows computations using Gröbner bases. Details on the types of algebras involved may be found in [2, 15].

Let $\mathcal{B}'$ be the subalgebra of $U_{r,s}(\mathfrak{sl}_3)$ generated by $\{f_1, f_2, \omega_1', \omega_2'\}$. Adding the element $\mathcal{F}_{21} = f_2 f_1 - s f_1 f_2$ to the generating set, and rewriting the relations of Definition 4.1, we see that $\mathcal{B}'$ is generated by

$\{x_1 = f_1,\ x_2 = \mathcal{F}_{21},\ x_3 = f_2,\ x_4 = \omega_1',\ x_5 = \omega_2'\}$, subject to relations

$$\{x_j x_i = C_{ij} x_i x_j + D_{ij} \mid 1 \le i < j \le 5\}$$

where the coefficients C_{ij} and elements D_{ij} are given as follows:

(1) $C_{12} = C_{23} = C_{25} = r$,
(2) $C_{13} = C_{15} = s$,
(3) $C_{24} = s^{-1}$,
(4) $C_{34} = r^{-1}$,
(5) $C_{14} = C_{35} = rs^{-1}$,
(6) $C_{45} = 1$,
(7) $D_{ij} = 0$ if $(i, j) \ne (1, 3)$ and $D_{13} = \mathcal{F}_{21}$.

Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{B}'$ generated by the set

$$\left\{(\omega_1')^\ell - 1,\, (\omega_2')^\ell - 1,\, f_1^\ell,\, \mathcal{F}_{21}^\ell,\, f_2^\ell\right\},$$

so that $H = H_{r,s} = (\mathfrak{b}'_{r,s})^{\mathrm{coop}} = \mathcal{B}'/\mathcal{I}$.

Equation (5.2) shows that if $g \in G(H)$, then $H\bullet_\beta g$ is spanned by

$$\left\{(f_1^k \mathcal{F}_{21}^t f_2^m)\bullet_\beta g \mid 0 \le k,\, t,\, m < \ell\right\}.$$

Applying equation (5.1) recursively, we may define a procedure `Beta` so that if $0 \le k, t, m < \ell$, $h \in H$, and $\beta : H \to \mathbb{K}$ is an algebra map given by $\beta(f_1) = q^a$ and $\beta(f_2) = q^b$, then `Beta(a,b,k,t,m,h)` gives $(f_1^k \mathcal{F}_{21}^t f_2^m)\bullet_\beta h$. Fix a grouplike element $g = (\omega_1')^{d_1}(\omega_2')^{d_2} \in H$. In what follows we will construct a basis and compute the dimension of the module $H\bullet_\beta g$. Some of the code for computing these bases, written by the second author, may be found in [19, 21]. Let

$$\mathcal{F}_\ell = \{f_1^k \mathcal{F}_{21}^t f_2^m \mid 0 \le k,\, t,\, m < \ell\}$$

(so that $H\bullet_\beta g = \mathsf{span}_{\mathbb{K}}\{f\bullet_\beta g \mid f \in \mathcal{F}_\ell\}$). Consider the linear map $T_\beta : \mathsf{span}_{\mathbb{K}}\mathcal{F}_\ell \to H$ given by $T_\beta(f) = f\bullet_\beta g$, and construct the matrix M representing T_β in the bases $\mathcal{F}_\ell$ and $\{fh \mid f \in \mathcal{F}_\ell, h \in G(H)\}$ of $\mathsf{span}_{\mathbb{K}}\mathcal{F}_\ell$ and H respectively. Then $\dim(H\bullet_\beta g) = \mathrm{rank}(M)$, and the nonzero columns of the column-reduced Gauss normal form of M give the coefficients for the elements of a basis of $H\bullet_\beta g$.

Note that $\dim(H) = \ell^5$ and $\dim(\mathsf{span}_{\mathbb{K}}\mathcal{F}_\ell) = \ell^3$, so the size of M is $\ell^5 \times \ell^3$. Computing the Gauss normal form of these matrices is an expensive calculation even for small values of ℓ such as $\ell = 5$. However, by some reordering of $\mathcal{F}_\ell$ and of the PBW basis of H, M is block diagonal. We explain this next.

For a monomial $m = f_1^{a_1}\mathcal{F}_{21}^{a_2} f_2^{a_3}(\omega_1')^{a_4}(\omega_2')^{a_5}$, let $\deg_1(m) = a_1 + a_2$ and $\deg_2(m) = a_2 + a_3$. Since $\mathcal{F}_{21} = f_2 f_1 - s f_1 f_2$ and (5.1) is homogeneous in f_1 and in f_2, $m\bullet_\beta x$ is a linear combination of monomials whose degrees are $\deg_i(m) + \deg_i(x)$. For all $0 \le u, v < 2\ell$, let

$$A_{(u,v)} = \{f \in \mathcal{F}_\ell \mid \deg_1(f) = u \text{ and } \deg_2(f) = v\}$$

and

$$B_{(u,v)} = \{f(\omega_1')^{-u}(\omega_2')^{-v} g \mid f \in A_{(u,v)}\}.$$

Then for all $f \in A_{(u,v)}$, we have $f\bullet_\beta g \in \mathsf{span}_{\mathbb{K}} B_{(u,v)}$. The possible pairs (u,v) are such that $0 \le u, v \le 2(\ell-1)$, and if $u > v$ (resp. $v > u$), then $|u - v|$ is the minimum power of f_1 (resp. f_2) that must be a factor of a monomial in $A_{(u,v)}$. Therefore $|u-v| \le \ell - 1$; that is, $u - (\ell-1) \le v \le u + \ell - 1$. Another way of describing the sets $A_{(u,v)}$ and $B_{(u,v)}$ is as follows.

$$\begin{aligned} A_{(u,v)} &= \{f_1^{u-i}\mathcal{F}_{21}^i f_2^{v-i} \mid i \in \mathbb{N} \text{ and } 0 \le u-i,\, i,\, v-i \le \ell - 1\} \\ &= \{f_1^{u-i}\mathcal{F}_{21}^i f_2^{v-i} \mid i \in \mathbb{N} \text{ and } \mathsf{n}_{u,v} \le i \le \mathsf{m}_{u,v}\}, \end{aligned}$$

where $\mathsf{n}_{u,v} = \max(0, \ell-1-u, \ell-1-v)$ and $\mathsf{m}_{u,v} = \min(\ell-1, u, v)$. Since $(\omega_i')^{-1} = (\omega_i')^{\ell-1}$, if $g = (\omega_1')^{d_1}(\omega_2')^{d_2}$ we also have

$$B_{(u,v)} = \{f(\omega_1')^{(\ell-1)u+d_1}(\omega_2')^{(\ell-1)v+d_2} \mid f \in A_{(u,v)}\}.$$

It is clear that $\mathcal{F}_\ell = \bigcup_{(u,v)} A_{(u,v)}$, a disjoint union, and $H \bullet_\beta g = \bigoplus_{(u,v)} \mathsf{span}_{\mathbb{K}} B_{(u,v)}$. Therefore the disjoint union of bases for $\big(\mathsf{span}_{\mathbb{K}} A_{(u,v)}\big) \bullet_\beta g$ for all possible pairs (u,v) gives a basis for $H \bullet_\beta g$, and thus $\dim(H \bullet_\beta g) = \sum_{(u,v)} \dim \big((\mathsf{span}_{\mathbb{K}} A_{(u,v)}) \bullet_\beta g\big)$.

Examples of computations

We now present just a few examples computed using these techniques; more examples may be found in [21]. These examples show that for some values of the parameters, we obtain a significantly different representation theory for the two-parameter quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$ than for any one-parameter quantum group $\mathfrak{u}_q(\mathfrak{sl}_n)$.

Example 5.6. Let q be a primitive 4th root of 1 and take $n = 3$. We compare the dimensions of the simple Yetter-Drinfeld modules for $H_{1,q}$ and $H_{q,q^{-1}}$. These Hopf algebras have the same dimension, but by Theorem 4.12, there is no 2-cocycle σ for which $(H_{1,q})^\sigma$ is isomorphic to $H_{q,q^{-1}}$. Considering the dimension sets below, we see that the representation theories of the two algebras are quite different. The results are displayed as multisets of dimensions, where each dimension is raised to the number of nonisomorphic simple Yetter-Drinfeld modules of that dimension.

$$\dim\left(H_{1,q} \bullet_\beta g\right),\ g \in G(H_{1,q}),\ \beta \in \widehat{G(H_{1,q})}: \quad \{1^{16}, 3^{32}, 6^{32}, 8^{16}, 10^{32}, 12^{32}, 24^{32}, 26^{16}, 42^{32}, 64^{16}\} \tag{5.7}$$

$$\dim\left(H_{q,q^{-1}} \bullet_\beta g\right),\ g \in G(H_{q,q^{-1}}),\ \beta \in \widehat{G(H_{q,q^{-1}})}: \qquad \{1^{16}, 3^{32}, 8^{16}, 16^{96}, 32^{96}\}. \tag{5.8}$$

If for some $\gamma \in \widehat{G(H)}$ and $g \in G(H)$, $H \bullet_\gamma g$ is one-dimensional, then $(H \bullet_{\gamma^{-1}} g^{-1}) \otimes (H \bullet_\gamma g) \cong H \bullet_\varepsilon 1 \cong H/\ker \varepsilon$, where $H \bullet_{\gamma^{-1}} g^{-1}$ is the one-dimensional module constructed from $g^{-1} \in G(H)$ (compare [20, p. 961]). Since $(H \bullet_\varepsilon 1) \otimes M \cong M$ for all modules $M \in {}_H\mathcal{YD}^H$, tensoring a simple Yetter-Drinfeld module with a one-dimensional module yields another simple Yetter-Drinfeld module of the same dimension. Because there are 16 one-dimensional modules for each of these algebras, this accounts for the fact that the superscripts in (5.7) and (5.8) are multiples of 16.

The remaining cases $H_{r,s}$, where r, s are 4th roots of 1 and $\mathrm{lcm}(|r|, |s|) = 4$, are all related to the ones in (5.7) and (5.8): By Theorem 4.12, H_{q,q^2}, H_{q^2,q^3}, and $H_{q^3,1}$ are all cocycle twists of $H_{1,q}$. Now q^3 is also a primitive 4th root of 1, and so the representation theory of H_{1,q^3} is the same as that of $H_{1,q}$, and cocycle twists of H_{1,q^3} are $H_{q,1}$, $H_{q^2,q}$, and H_{q^3,q^2}. Finally, $H_{q^3,q}$ is a cocycle twist of H_{q,q^3}. In fact, it follows from the quantum group isomorphisms in [12, Thm. 5.5], that $H_{1,q} \cong H_{q^3,1}$, $H_{q,1} \cong H_{1,q^3}$, $H_{q^2,q} \cong H_{q^3,q^2}$, and $H_{q,q^2} \cong H_{q^2,q^3}$.

Example 5.9 (Dimensions of simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules)**.** Assume $n = 3$, $\gcd(6, \ell) = 1$, $\mathrm{lcm}(|r|, |s|) = \ell$, rs^{-1} is a primitive ℓth root of 1, and q is a primitive ℓth root of 1 such that $q^2 = rs^{-1}$. By Theorem 4.12, $H^\sigma_{r,s} \cong H_{q,q^{-1}}$. The dimensions of the simple Yetter-Drinfeld $H_{q,q^{-1}}$-modules are given by the dimensions of simple $\mathfrak{u}_q(\mathfrak{sl}_3)$-modules; the latter can be found in [8]. Hence, this gives formulas for the dimensions of the simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules when it is a Drinfeld double. If it is not a Drinfeld double, we just get the dimensions for the simple Yetter-Drinfeld $H_{r,s}$-modules and for the simple modules of its double $D(H_{r,s})$.

Write $r = q^y$ and $s = q^z$. Let $g = \omega_1'^{d_1}\omega_2'^{d_2} \in G$ and $\beta : \mathbb{K}G \to \mathbb{K}$ be an algebra map given by $\beta(\omega_i') = q^{\beta_i}$. Taking $\sigma = \sigma_{r,q}$ (see the proof of Theorem 4.12), we have $(H_{r,s}\bullet_\beta g)^\sigma \cong H_{q,q^{-1}}\bullet_\gamma g$ by Theorems 3.9 and 4.12 and (4.16), where $\gamma(\omega_i') = q^{\gamma_i}$ with

$$\gamma_1 = \beta_1 - d_2(y-1) \quad \text{and} \quad \gamma_2 = \beta_2 + d_1(y-1).$$

Since $\gcd(6,\ell) = 1$, we have $H_{q,q^{-1}}\bullet_\gamma g \cong M \otimes N$ with M a simple $\mathfrak{u}_q(\mathfrak{sl}_3)$-module and N a one-dimensional $\mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_3)$-module by [20, Thm. 2.12] (compare Remark 4.19). In particular, $\dim(H_{r,s}\bullet_\beta g) = \dim\big(H_{q,q^{-1}}\bullet_\gamma g\big) = \dim(M)$. By the proof of [20, Thm. 2.12], $M = H_{q,q^{-1}}\bullet_{\beta_h} h$, where $h = \omega_1'^{c_1}\omega_2'^{c_2}$, (c_1, c_2) a solution of the following system of equations in $\mathbb{Z}/\ell\mathbb{Z}$,

$$\begin{pmatrix} \mathrm{Id} & \mathsf{A}^{-1} \\ -\mathsf{A} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \gamma_1 \\ \gamma_2 \end{pmatrix},$$

and A is the Cartan matrix

$$\mathsf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix};$$

β_h is defined by $\beta_h(\omega_1') = q^{c_2 - 2c_1}$, $\beta_h(\omega_2') = q^{c_1 - 2c_2}$; and the one-dimensional module N is given by the character χ of $G(H_{q,q^{-1}})$ specified by $\chi(\omega_1') = q^{\chi_1}$, $\chi(\omega_2') = q^{\chi_2}$.

Let m_1 and m_2 be defined by

$$m_1 \equiv (2c_1 - c_2 + 1) \bmod \ell, \quad m_2 \equiv (2c_2 - c_1 + 1) \bmod \ell \quad \text{and} \quad 0 < m_i \le \ell.$$

By [8] (see the summary at [21]) and the above discussion, we have:

- If $m_1 + m_2 \le \ell$, then
$$\dim(H_{r,s}\bullet_\beta g) = \frac{1}{2}\Big(m_1 m_2 (m_1 + m_2)\Big).$$
- If $m_1 + m_2 > \ell$ and $m_i' = \ell - m_i$, then
$$\dim(H_{r,s}\bullet_\beta g) = \frac{1}{2}\Big(m_1 m_2 (m_1 + m_2)\Big) - \frac{1}{2}\Big(m_1' m_2' (m_1' + m_2')\Big).$$

The parameters m_1 and m_2 can be obtained from the original data:

$$m_1 \equiv (d_1 - d_2 + \frac{1}{2}(d_2 y - \beta_1) + 1) \bmod \ell, \quad m_2 \equiv (d_2 - \frac{1}{2}(y d_1 + \beta_2) + 1) \bmod \ell. \tag{5.10}$$

Remark 5.11. Suppose in addition to the assumptions of Example 5.9 that hypothesis (4.5) also holds. Let G_C denote the set of central grouplike elements of $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$, and let $\mathfrak{J}$ be the ideal of $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$ generated by $\{h - 1 \mid h \in G_C\}$. It was shown in [20, Prop. 2.7] that the simple modules for the quotient $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_3)} = \mathfrak{u}_{r,s}(\mathfrak{sl}_3)/\mathfrak{J}$ are of the form $H\bullet_{\beta_g} g$, where if $g = \omega_1'^{d_1}\omega_2'^{d_2}$, then $\beta_g(\omega_i') = q^{\beta_i}$ with $\beta_1 = (z-y)d_1 + d_2 y$ and $\beta_2 = -z d_1 + (z-y)d_2$. Using the formulas in (5.10) and the fact that $y - z \equiv 2 \mod \ell$ (since $q^2 = rs^{-1}$), we get

$$m_1 \equiv (2d_1 - d_2 + 1) \bmod \ell, \qquad m_2 \equiv (2d_2 - d_1 + 1) \bmod \ell.$$

Thus, for $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_3)}$-modules, the dimensions obtained in Example 5.9 are those conjectured in [19, Conjecture III.4]. Furthermore, under these hypotheses, every simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-module is a tensor product of a one-dimensional module with a simple $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_3)}$-module by [20, Thm. 2.12]. Thus, in order to determine all possible dimensions of simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules, it is enough to use the formulas in Example 5.9 with $m_1 \equiv (2d_1 - d_2 + 1) \mathrm{mod}\ \ell$ and $m_2 \equiv (2d_2 - d_1 + 1) \bmod \ell$, for all $d_1, d_2 \in \{0, 1, \cdots, \ell - 1\}$.

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